# Mathematical Background of Formalism of Operator Manifold

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September 3, 1997



**Abstract**

The analysis of mathematical structure of the method of operator manifold [1] guides our discussion. The nature of operator manifold provides its elements with both quantum field and differential geometry aspects, a detailed study of which is a subject of present paper. It yields a quantization of geometry differing in principle from all earlier suggested schemes. This formalism was made complete by construction of state wave functions and calculation of matrix elements of geometric objects. While, it has been shown that the matrix element of any geometric object of operator manifold gave rise to corresponding geometric object of wave manifold.

## 1 Introduction

In spite of considerable progress achieved over the entire subsequent period in the study of fundamental constituents of matter and forces, the physical theory is still far from being complete and could not be regarded as the final word in particle physics, since many fundamental questions have yet to be answered. The absence of the vital theory which will be able to solve the crucial problems of particle physics imperatively stimulates a search for general constructive principles. Effecting a reconciliation in [1] we are led to consider the theory exploring the query of origin of geometry, some basic concepts of particle physics and also four major principles of Relativity, Quantum, Gauge and Color Confinement. We have finally arrived at the scenario a whole idea of which comes to following: the geometry, quarks with various quantum numbers, internal symmetries and etc. also those four principles, as it was proven, are derivative and come into being simultaneously.

In [1] we elaborated a new mathematical framework, which is a still wider generalization of the method of secondary quantization with appropriate expansion over the geometric objects. The formalism of *operator manifold* ensued, which is a guiding formalism framing our approach yielding a quantization of geometry differing in principle from all earlier suggested schemes. The nature of operator manifold provides its elements with both field and geometric aspects. To save writing, in [1] they have been discussed briefly, leaving a more profound mathematical study for separate treatment. As far as suggested theory involves a drastic revision of our ideas of geometry, basic concepts and principles of particle physics, in order to be more consistent and convinced in correctness of drawn statements it will be advantageous to fill this shortage and verify them by further exploration of structure of the theory. There is an attempt to supply the mathematical background and to trace some of the major currents of thoughts under its view-point. In view of all this, the detailed study of different aspects surely an important subject for present article. It

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can be regarded as a mathematical addition to [1]. The article is organized as follows: To begin with our task will be to make few preliminary remarks on the structure of manifold $G(2.2.3)$, on the base of which we define general conceptual part of formalism of operator manifold $\hat{G}(2.2.3)$. In the aftermath, we analyze the mathematical structure of the method of $\hat{G}(2.2.3)$ in both aspects of quantum field theory and differential geometry. This is not a final report on a closed subject, but it is hoped that suggested theory will serve as useful introduction and that it will thereby add a knowledge of method of operator manifold in quantization of geometry.

# 2 Operator-Vector and Co-Vector Fields

Before embarking on the main strategy of discussion, as a starting point, just a very brief recapitulation of major properties of structure of manifold $G(2.2.3)$ [1-6]. Let the maximal curve $\lambda(t) : R^1 \to G(2.2.3)$ passed through the point $p = \lambda(0) \in G(2.2.3)$ with tangent vector $\mathbf{A}|_{\lambda(t)}$, where the $G(2.2.3)$ is 12-dimensional smooth differentiable manifold. The set $\{\zeta^{(\lambda,\mu,\alpha)}\}$ $(\lambda, \mu = 12; \quad \alpha = 1, 2, 3)$ are local coordinates in open neighborhood of $p \in \mathcal{U}$, namely the curve $\lambda(t)$ has the coordinates $\zeta^{(\lambda,\mu,\alpha)}(t)$. For any other point $q \in G(2.2.3)$ it can be found open neighborhood $\mathcal{U}$ and $\varepsilon > 0$ such that $\mathbf{A}$ gives the diffeomorphism $f_t : \mathcal{U} \to G(2.2.3)$ at $|t| < \varepsilon$. The 12-dimensional smooth vector field $\mathbf{A}_p = \mathbf{A}(\zeta)$ belongs to the section of tangent bundle $\mathbf{T}_p$ at the point $p(\zeta)$. So define the differential $d\,A^t_p$ of the flux $A^t_p : G(2.2.3) \to G(2.2.3)$ at the point $p(\zeta) \in G(2.2.3)$ with the field of velocities $\mathbf{A}(\zeta)$, then one gets $d\,A^t_p : \mathbf{T}_p \to R$, namely the one-parameter group of diffeomorphismes $A^t$ given for the maximal curve $\zeta(t)$ passing through point $p$ and $\zeta(0) = \zeta_p, \quad \dot{\zeta}(0) = \mathbf{A}_p$

$$d\,A^t_p(\mathbf{A}) = \frac{d}{d\,t}\bigg|_{t=0} A^t\,(\zeta(t)) = \mathbf{A}_p(\zeta). \tag{2.1}$$

Hence,

$$d\,A^t : \mathbf{T}(G(2.2.3)) \to R \left( \mathbf{T}(G(2.2.3)) = \bigcup_{p(\zeta)} \mathbf{T}_p \right).$$

Let the $\{e_{(\lambda,\mu,\alpha)} = O_{\lambda,\mu} \otimes \sigma_\alpha\} \subset G(2.2.3)$ is some set of linear independent 12 unit vectors at the point $p$, provided with the linear unit bi-pseudo-vectors $\{O_{\lambda,\mu}\}$: $< O_{\lambda,\mu}, O_{\tau,\nu} >= {}^*\delta_{\lambda,\tau}{}^*\delta_{\mu,\nu}$ serving as the basis for tangent vectors of $2 \times 2$ dimensional linear bi-pseudo-space ${}^*G(2.2)$, ${}^*\delta_{\lambda,\tau} = 1 - \delta_{\lambda,\tau}$, where $\delta$ is Kronecker symbol, ; and the ordinary unit vectors $\sigma_\alpha$ implying $< \sigma_\alpha, \sigma_\beta >= \delta_{\alpha\beta}$. Henceforth we always let the first two subscripts in the parentheses specify the bi-pseudo-vector components, while the third refers to the ordinary-vector components. The metric on $G(2.2.3)$ is bilinear, local, symmetric and positive defined reflection of vector fields of sections $\mathbf{T}$ of tangent bundle of $G(2.2.3)$, namely $\hat{\mathbf{g}} : \mathbf{T}_p \otimes \mathbf{T}_p \to C^\infty(G(2.2.3))$ a section of conjugate vector bundle $S^2\mathbf{T}$ (symmetric part of tensor degree) with components in basis $\{e_{(\lambda,\mu,\alpha)}\}$

$$g_{(\lambda,\mu,\alpha)(\tau,\nu,\beta)} = g(e_{(\lambda,\mu,\alpha)}, e_{(\tau,\nu,\beta)}) = g(e_{(\tau,\nu,\beta)}, e_{(\lambda,\mu,\alpha)}). \tag{2.2}$$

Any vector $\mathbf{A}_p \in \mathbf{T}_p$ reads $\mathbf{A} = e_{(\lambda,\mu,\alpha)}A^{(\lambda,\mu,\alpha)}$, provided with components $A^{(\lambda,\mu,\alpha)}$ in the basis $\{e_{(\lambda,\mu,\alpha)}\}$. Except where stated otherwise, here as usual, the double occurrence of

the dummy indices will be taken to denote a summation extended over their all values. In holonomic coordinate basis $\left(\partial/\partial\,\zeta^{(\lambda,\mu,\alpha)}\right)_p$ one gets $A^{(\lambda,\mu,\alpha)} = \left.\dfrac{d\,\zeta^{(\lambda,\mu,\alpha)}}{d\,t}\right|_p$ and $\hat{g} = g_{(\lambda,\mu,\alpha)(\tau,\nu,\beta)}d\zeta^{(\lambda,\mu,\alpha)} \otimes d\zeta^{(\tau,\nu,\beta)}$. The manifold $G(2.2.3)$ decomposes as follows: $G(2.2.3) = {}^*\mathbf{R}^2 \otimes {}^*\mathbf{R}^2 \otimes \mathbf{R}^3 = \underset{\eta}{G}(2.3) \oplus \underset{u}{G}(2.3) = \sum_{\lambda,\mu=1}^{2} \oplus \mathbf{R}^3_{\lambda\mu} = \underset{x}{\mathbf{R}}^3 \oplus \underset{x}{T}^3 \oplus \underset{u}{\mathbf{R}}^3 \oplus \underset{u}{T}^3$ with corresponding basis vectors $\underset{i\,(\lambda\alpha)}{e^0} = \underset{i\;\lambda}{O} \otimes \sigma_\alpha \subset \underset{i}{G}(2.3)$ $(i = \eta, u)$ of tangent sections, where $\underset{i\;+}{O} = \dfrac{1}{\sqrt{2}}(O_{1,1} + \varepsilon_i O_{2,1})$, $\underset{i\;-}{O} = \dfrac{1}{\sqrt{2}}(O_{1,2} + \varepsilon_i O_{2,2})$, $\varepsilon_\eta = 1$, $\varepsilon_u = -1$. There up on $< \underset{i\;\lambda}{O}, \underset{i\;\tau}{O} >= \varepsilon_i \delta_{ij} {}^*\delta_{\lambda\tau}$. The positive metric forms are defined in manifolds $\underset{i}{G}(2.3)$ $\eta^2 = \eta_{(\lambda\alpha)}\eta^{(\lambda\alpha)} \in \underset{\eta}{G}(2.3)$, $\quad u^2 = u_{(\lambda\alpha)}u^{(\lambda\alpha)} \in \underset{u}{G}(2.3)$, where

$$\begin{aligned}
\eta^{(+\alpha)} &= \frac{1}{\sqrt{2}}(\zeta^{(1,1,\alpha)} + \zeta^{(2,1,\alpha)}), \quad \eta^{(-\alpha)} = \frac{1}{\sqrt{2}}(\zeta^{(1,2,\alpha)} + \zeta^{(2,2,\alpha)}),\\
u^{(+\alpha)} &= \frac{1}{\sqrt{2}}(\zeta^{(1,1,\alpha)} - \zeta^{(2,1,\alpha)}), \quad u^{(-\alpha)} = \frac{1}{\sqrt{2}}(\zeta^{(1,2,\alpha)} - \zeta^{(2,2,\alpha)}).
\end{aligned}$$

The $G(2.3)$ decomposes into three-dimensional ordinary ($\mathbf{R}^3$) and time ($\mathbf{T}^3$) flat spaces $G(2.3) = \mathbf{R}^3 \oplus \mathbf{T}^3$ with signatures $sgn(\mathbf{R}^3) = (+++)$ and $sgn(\mathbf{T}^3) = (---)$. Since all directions in $\mathbf{T}^3$ are equivalent, then by notion *time* one implies the projection of time-coordinate on fixed arbitrary universal direction in $\mathbf{T}^3$. By the reduction $\mathbf{T}^3 \to T^1$ the transition $G(2.3) \to M^4 = \mathbf{R}^3 \oplus T^1$ may be performed whenever it will be needed. At this point we cut short a discussion of structure of $G(2.2.3)$ and refer to [3,4] for details. Next we proceed to preliminary definitions of the elements of operator manifold, which will be further discussed and made complete in due course of exposition of the formalism describing the processes of creation and annihilation of geometric objects. This formalism is analogous to the method of secondary quantization describing the processes of creation and annihilation of particles in the configuration space of occupation numbers, but it will be an appropriate expansion over the geometric objects. Adjusting to fit a conventional notations, below we change the order of vector's and co-vector's indices used in [1] into the opposite. So, the state of $\zeta$-type quantum is describe by means of function $\Phi^{(\lambda,\mu,\alpha)}(\zeta)$ belonging to the ring of functions of $C^\infty$-class:

$$\begin{aligned}
\Phi^{(1,1,\alpha)} &= \frac{1}{\sqrt{2}}\left(\underset{\eta}{\Psi}^{(+\alpha)} + \underset{u}{\Psi}^{(+\alpha)}\right), \quad \Phi^{(1,2,\alpha)} = \frac{1}{\sqrt{2}}\left(\underset{\eta}{\Psi}^{(-\alpha)} + \underset{u}{\Psi}^{(-\alpha)}\right),\\
\Phi^{(2,1,\alpha)} &= \frac{1}{\sqrt{2}}\left(\underset{\eta}{\Psi}^{(+\alpha)} + \underset{u}{\Psi}^{(+\alpha)}\right), \quad \Phi^{(2,2,\alpha)} = \frac{1}{\sqrt{2}}\left(\underset{\eta}{\Psi}^{(-\alpha)} + \underset{u}{\Psi}^{(-\alpha)}\right),
\end{aligned}$$

provided by the functions of $\eta$- and $u$-type quanta defined on the manifolds $\underset{i}{G}(2.3)$

$$\begin{aligned}
\underset{\eta}{\Psi}^{(\pm\alpha)}(\eta, p_\eta) &= \eta^{(\pm\alpha)}\underset{\eta}{\Psi}^{\pm}(\eta, p_\eta),\\
\underset{u}{\Psi}^{(\pm\alpha)}(u, p_u) &= u^{(\pm\alpha)}\underset{u}{\Psi}^{\pm}(u, p_u).
\end{aligned} \tag{2.3}$$

Here it was assumed that the probability of finding the quantum in the state with fixed coordinate ($\eta$ or $u$) and momentum ($p_\eta$ or $p_u$) is determined by the square of its state wave function $\underset{\eta\,\pm}{\Psi}(\eta, p_\eta)$, or $\underset{u\,\pm}{\Psi}(u, p_u)$. This provides a simple intuitive meaning of state

functions, where the 6-vectors of coordinates - $\eta, u$, and momenta - $p_\eta, p_u$ respectively are $\eta = \underset{\eta(\lambda\alpha)}{e^0}\eta^{(\lambda\alpha)}, \quad p_\eta = \underset{\eta(\lambda\alpha)}{e^0}\underset{\eta}{p^{(\lambda\alpha)}}, \quad u = \underset{u(\lambda\alpha)}{e^0}u^{(\lambda\alpha)}, \quad p_u = \underset{u(\lambda\alpha)}{e^0}\underset{u}{p^{(\lambda\alpha)}}.$
Being confronted by the problem of quantization of geometry, we first deal with a substitution of the basis elements by the corresponding operators of creation and annihilation of quanta acting in the configuration space of occupation numbers. Instead of pseudo-vectors $O_\lambda$ we introduce the following operators supplied by additional index $(r)$ referring to the quantum numbers of corresponding state

$$\hat{O}^r_1 = O^r_1\alpha_1, \quad \hat{O}^r_2 = O^r_2\alpha_2, \quad \hat{O}^\lambda_r = {}^*\delta^{\lambda\mu}\hat{O}^r_\mu = (\hat{O}^r_\lambda)^+,$$
$$\{\hat{O}^r_\lambda, \hat{O}^{r'}_\tau\} = \delta_{rr'}{}^*\delta_{\lambda\tau}I_2, \quad < O^r_\lambda, O^{r'}_\tau >= \delta_{rr'}{}^*\delta_{\lambda\tau}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.4}$$

The matrices $\alpha_\lambda$ satisfy the conditions

$$\alpha^\lambda = {}^*\delta^{\lambda\mu}\alpha_\mu = (\alpha_\lambda)^+, \qquad \{\alpha_\lambda, \alpha_\tau\} = {}^*\delta_{\lambda\tau}I_2. \tag{2.5}$$

For example, they may be in the form $\alpha_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This forms the starting point for quantization. Creation operator $\hat{O}^r_1$ generates one-occupied state $\mid 1 >_{(0)} \equiv \mid 0, \ldots, 1, \ldots >_{(0)}$ and the basis vector $O^r_1$ with the quantum number $r$ right through acting on non-occupied vacuum state $\mid 0 >_{(0)} \equiv \mid 0, 0, \ldots >_{(0)}$:

$$\hat{O}^r_1 \mid 0 >_{(0)} = O^r_1 \mid 1 >_{(0)} . \tag{2.6}$$

Accordingly, the action of annihilation operator $\hat{O}^r_2$ on one-occupied state yields the vacuum state and the basis vector $O^r_2$

$$\hat{O}^r_2 \mid 1 >_{(0)} = O^r_2 \mid 0 >_{(0)} . \tag{2.7}$$

So define $\hat{O}^r_1 \mid 1 >_{(0)} = 0, \quad \hat{O}^r_2 \mid 0 >_{(0)} = 0$. The matrix realization of the states $\mid 0 >_0$ and $\mid 1 >_0$, for instance, may be as follows: $\mid 0 >_0 \equiv \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mid 1 >_0 \equiv \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
The operator of occupation number is $\hat{N}^{(0)}_r = \hat{O}^r_1\hat{O}^r_2$, with the expectation values implying Pauli's exclusion principle ${}_{(0)} < 0 \mid \hat{N}^{(0)}_r \mid 0 >_{(0)} = 0, \qquad {}_{(0)} < 1 \mid \hat{N}^{(0)}_r \mid 1 >_{(0)} = 1$. The vacuum state reads $\chi_0 \equiv \mid 0 >_{(0)} = \prod_{r=1}^{N}(\chi_1)_r$. With this final detail cared for one-occupied state takes the form $\chi_{r'} \equiv \mid 1 >_{(0)} = (\chi_2)_{r'}\prod_{r \neq r'}(\chi_1)_r$. Continuing along this line, instead of ordinary vectors we introduce the operators $\hat{\sigma}^r_\alpha \equiv \delta_{\alpha\beta\gamma}\sigma^r_\beta\widetilde{\sigma}_\gamma$, where $\widetilde{\sigma}_\gamma$ are Pauli's matrices, and

$$< \sigma^r_\alpha, \sigma^{r'}_\beta >= \delta_{rr'}\delta_{\alpha\beta}, \quad \hat{\sigma}^\alpha_r = \delta^{\alpha\beta}\hat{\sigma}^r_\beta = (\hat{\sigma}^r_\alpha)^+ = \hat{\sigma}^r_\alpha, \quad \{\hat{\sigma}^r_\alpha, \hat{\sigma}^{r'}_\beta\} = 2\delta_{rr'}\delta_{\alpha\beta}I_2. \tag{2.8}$$

For the vacuum state $\mid 0 >_{(\sigma)} \equiv \varphi_{1(\alpha)}$ and one-occupied state $\mid 1_{(\alpha)} >_{(\sigma)} \equiv \varphi_{2(\alpha)}$ we make use of matrix realization $\varphi_{1(\alpha)} \equiv \chi_1, \quad \varphi_{2(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_{2(2)} = \begin{pmatrix} -i \\ 0 \end{pmatrix}, \quad \varphi_{2(3)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
Then

$$\hat{\sigma}^r_\alpha\varphi_{1(\alpha)} = \sigma^r_\alpha\varphi_{2(\alpha)} = (\sigma^r_\alpha\widetilde{\sigma}_\alpha)\varphi_{1(\alpha)}, \quad \hat{\sigma}^r_\alpha\varphi_{2(\alpha)} = \sigma^r_\alpha\varphi_{1(\alpha)} = (\sigma^r_\alpha\widetilde{\sigma}_\alpha)\varphi_{2(\alpha)}. \tag{2.9}$$

Hence, the single eigen-value $(\sigma^r_\alpha \widetilde{\sigma}_\alpha)$ has associated with it quite different $\varphi_{\lambda(\alpha)}$. The eigen-value is degenerated with degeneracy degree equal 2. Due to it, along many quantum numbers $r$ there is also the quantum number of the spin $\vec{\sigma}$ with the values $\sigma_3 = \frac{1}{2}s \quad (s = \pm 1)$. This rule for spin quantum number is not without an important reason. The argument for this conclusion is compulsory suggested by the properties of operators $\hat{\sigma}^r_\alpha$. As it was seen in [1], this consequently *gives rise to spin of particle.*
One-occupied state reads $\varphi_{r'(\alpha)} = (\varphi_{2(\alpha)})_{r'} \prod_{r \neq r'} (\chi_1)_r$. Next we introduce the operators

$$\hat{\gamma}^r_{(\lambda,\mu,\alpha)} \equiv \hat{O}^{r_1}_\lambda \otimes \hat{O}^{r_2}_\mu \otimes \hat{\sigma}^{r_3}_\alpha, \quad \hat{\gamma}^{(\lambda,\mu,\alpha)}_r \equiv \hat{O}^\lambda_{r_1} \otimes \hat{O}^\mu_{r_2} \otimes \hat{\sigma}^\alpha_{r_3} = {}^*\delta^{\lambda\tau *}\delta^{\mu\nu}\delta^{\alpha\beta}\hat{\gamma}^r_{(\tau,\nu,\beta)}, \tag{2.10}$$

and also the state vector

$$\chi_{\lambda,\mu,\tau(\alpha)} \equiv \mid \lambda,\mu,\tau(\alpha) >= \chi_\lambda \otimes \chi_\mu \otimes \varphi_{\tau(\alpha)}, \tag{2.11}$$

where $\lambda,\mu,\tau,\nu = 1,2; \quad \alpha,\beta = 1,2,3$ and $r \equiv (r_1,r_2,r_3)$. Hence $\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\chi_{\tau,\nu,\delta(\beta)} = (\hat{O}^{r_1}_\lambda \chi_\tau) \otimes (\hat{O}^{r_2}_\mu \chi_\nu) \otimes (\hat{\sigma}^{r_3}_\alpha \varphi_{\delta(\beta)})$. Omitting the two-valuedness of state vector we apply $\mid \lambda,\tau,\delta(\beta) > \equiv \mid \lambda,\tau >$, and the same time remember that always the summation must be extended over the double degeneracy of the spin states $(s = \pm 1)$.
With this final detail cared for one infers the explicit forms of corresponding matrix elements:

$$\begin{array}{ll}
< 2,2 \mid \hat{\gamma}^r_{(1,1,\alpha)} \mid 1,1 >= e^r_{(1,1,\alpha)}, & < 1,1 \mid \hat{\gamma}^{(1,1,\alpha)}_r \mid 2,2 >= e^{(1,1,\alpha)}_r, \\
< 2,1 \mid \hat{\gamma}^r_{(1,2,\alpha)} \mid 1,2 >= e^r_{(1,2,\alpha)}, & < 1,2 \mid \hat{\gamma}^{(1,2,\alpha)}_r \mid 2,1 >= e^{(1,2,\alpha)}_r, \\
< 1,2 \mid \hat{\gamma}^r_{(2,1,\alpha)} \mid 2,1 >= e^r_{(2,1,\alpha)}, & < 2,1 \mid \hat{\gamma}^{(2,1,\alpha)}_r \mid 1,2 >= e^{(2,1,\alpha)}_r, \\
< 1,1 \mid \hat{\gamma}^r_{(2,2,\alpha)} \mid 2,2 >= e^r_{(2,2,\alpha)}, & < 2,2 \mid \hat{\gamma}^{(2,2,\alpha)}_r \mid 1,1 >= e^{(2,2,\alpha)}_r.
\end{array} \tag{2.12}$$

The operators of occupation numbers are

$$\begin{array}{l}
\underset{1}{\hat{N}}{}^{rr'}_{\alpha\beta} = \hat{\gamma}^r_{(1,1,\alpha)}\hat{\gamma}^{r'}_{(2,2,\beta)} = \underset{1}{\hat{N}}{}^{\alpha\beta}_{rr'} = \hat{\gamma}^{(2,2,\alpha)}_r \hat{\gamma}^{(1,1,\beta)}_{r'}, \\
\underset{2}{\hat{N}}{}^{rr'}_{\alpha\beta} = \hat{\gamma}^r_{(2,1,\alpha)}\hat{\gamma}^{r'}_{(1,2,\beta)} = \underset{2}{\hat{N}}{}^{\alpha\beta}_{rr'} = \hat{\gamma}^{(1,2,\alpha)}_r \hat{\gamma}^{(2,1,\beta)}_{r'},
\end{array} \tag{2.13}$$

with the expectation values implying Pauli's exclusion principle

$$\begin{array}{ll}
< 2,2 \mid \underset{1}{\hat{N}}{}^{\alpha\beta}_{rr'} \mid 2,2 >= \delta_{rr'}\delta_{\alpha\beta}, & < 1,2 \mid \underset{2}{\hat{N}}{}^{\alpha\beta}_{rr'} \mid 1,2 >= \delta_{rr'}\delta_{\alpha\beta}, \\
< 1,1 \mid \underset{1}{\hat{N}}{}^{\alpha\beta}_{rr'} \mid 1,1 >= 0, & < 2,1 \mid \underset{2}{\hat{N}}{}^{\alpha\beta}_{rr'} \mid 2,1 >= 0.
\end{array} \tag{2.14}$$

The set of operators $\{\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\}$ is the basis for operator-vectors $\hat{\Phi}(\zeta) = \hat{\gamma}^r_{(\lambda,\mu,\alpha)}\Phi^{(\lambda,\mu,\alpha)}_r(\zeta)$, but a set of operators $\{\hat{\gamma}^{(\lambda,\mu,\alpha)}_r\}$ is a dual basis for operator-co-vectors $\hat{\bar{\Phi}}(\zeta) = \hat{\gamma}^{(\lambda,\mu,\alpha)}_r \Phi^r_{(\lambda,\mu,\alpha)}(\zeta)$, where $\Phi^r_{(\lambda,\mu,\alpha)}(\zeta) = \bar{\Phi}^{(\lambda,\mu,\alpha)}_r(\zeta)$ (charge-conjugated). One easily gets

$$\begin{array}{l}
< 2,2 \mid \hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta) \mid 2,2 >= \Phi^{(1,1,\alpha)}_r(\zeta)\Phi^r_{(1,1,\alpha)}(\zeta), \\
< 2,1 \mid \hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta) \mid 2,1 >= \Phi^{(1,2,\alpha)}_r(\zeta)\Phi^r_{(1,2,\alpha)}(\zeta), \\
< 1,2 \mid \hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta) \mid 1,2 >= \Phi^{(2,1,\alpha)}_r(\zeta)\Phi^r_{(2,1,\alpha)}(\zeta), \\
< 1,1 \mid \hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta) \mid 1,1 >= \Phi^{(2,2,\alpha)}_r(\zeta)\Phi^r_{(2,2,\alpha)}(\zeta).
\end{array} \tag{2.15}$$

Introducing the state vectors

$$\chi^0(\nu_1,\nu_2,\nu_3,\nu_4) =\mid 1,1>^{\nu_1}\cdot\mid 1,2>^{\nu_2}\cdot\mid 2,1>^{\nu_3}\cdot\mid 2,2>^{\nu_4},$$

$$\nu_i = \begin{cases} 1 & \text{if } \nu=\nu_i \quad \text{for some } i, \\ 0 & \text{otherwise}, \end{cases} \tag{2.16}$$

$$\mid\chi_-(\lambda)>= \begin{cases} \chi^0(1,0,0,0) & \lambda=1, \\ \chi^0(0,0,1,0) & \lambda=2, \end{cases} \qquad \mid\chi_+(\lambda)>= \begin{cases} \chi^0(0,0,0,1) & \lambda=1, \\ \chi^0(0,1,0,0) & \lambda=2, \end{cases}$$

provided

$$\begin{aligned} &<\lambda,\mu,\mid\tau,\nu>=\delta_{\lambda\tau}\delta_{\mu\nu}, \quad <\chi_\pm\mid A\mid\chi_\mp>=\sum_\lambda<\chi_\pm(\lambda)\mid A\mid\chi_\mp(\lambda)>, \\ &<\chi_\pm(\lambda)\mid\chi_\pm(\mu)>=\delta_{\lambda\mu}, \quad <\chi_\pm\mid A\mid\chi_\pm>=\sum_\lambda<\chi_\pm(\lambda)\mid A\mid\chi_\pm(\lambda)>, \\ &<\chi_\pm(\lambda)\mid\chi_\mp(\mu)>=0, \end{aligned}$$

we get the matrix elements as follows:

$$\begin{aligned} <\chi_+\mid\hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta)\mid\chi_+>\equiv\Phi^2_+(\zeta) &= \quad \Phi^{(1,1,\alpha)}_r(\zeta)\Phi^r_{(1,1,\alpha)}(\zeta)+\Phi^{(2,1,\alpha)}_r(\zeta)\Phi^r_{(2,1,\alpha)}(\zeta), \\ <\chi_-\mid\hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta)\mid\chi_->\equiv\Phi^2_-(\zeta) &= \Phi^{(2,2,\alpha)}_r(\zeta)\Phi^r_{(2,2,\alpha)}(\zeta)+\Phi^{(1,2,\alpha)}_r(\zeta)\Phi^r_{(1,2,\alpha)}(\zeta). \end{aligned} \tag{2.17}$$

The basis $\{\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\}$ decomposes into $\{\underset{i}{\hat{\gamma}}^r_{(\lambda\alpha)}\}$ $\quad(\lambda=\pm;\quad \alpha=1,2,3;\quad i=\eta,u)$. The latter reads in component form $\underset{i}{\hat{\gamma}}^r_{(+\alpha)}=\dfrac{1}{\sqrt{2}}(\hat{\gamma}^r_{(1,1\alpha)}+\varepsilon_i\hat{\gamma}^r_{(2,1\alpha)}),\quad \underset{i}{\hat{\gamma}}^r_{(-\alpha)}=\dfrac{1}{\sqrt{2}}(\hat{\gamma}^r_{(1,2\alpha)}+\varepsilon_i\hat{\gamma}^r_{(2,2\alpha)})$. The expansions of operator-vectors $\underset{i}{\hat{\Psi}}\in\underset{i}{\hat{G}}(2.3)$ and operator-co-vectors $\underset{i}{\hat{\bar{\Psi}}}\in\underset{i}{\hat{G}}(2.3)$ are written $\underset{i}{\hat{\Psi}}=\underset{i}{\hat{\gamma}}^r_{(\lambda\alpha)}\underset{i}{\Psi}^{(\lambda\alpha)}_r,\quad \underset{i}{\hat{\bar{\Psi}}}=\underset{i}{\hat{\gamma}}^{(\lambda\alpha)}_r\underset{i}{\Psi}^r_{(\lambda\alpha)}$, where the components $\underset{\eta}{\Psi}^{(\lambda\alpha)}_r(\eta)$ and $\underset{u}{\Psi}^{(\lambda\alpha)}_r(u)$ are in the form eq.(2.3), and $\underset{i}{\Psi}^r_{(\lambda\alpha)}=\underset{i}{\bar{\Psi}}^{(\lambda\alpha)}_r$. The operator of occupation number of $i$-type quantum takes the form $\underset{i}{\hat{N}}^{\alpha\beta}_{rr'}=\varepsilon_i\underset{i}{\hat{\gamma}}^{(-\alpha)}_r\underset{i}{\hat{\gamma}}^{(+\beta)}_{r'}$, with corresponding expectation values

$$<\chi_-\mid\underset{i}{\hat{N}}^{\alpha\beta}_{rr'}\mid\chi_->=0, \quad <\chi_+\mid\underset{i}{\hat{N}}^{\alpha\beta}_{rr'}\mid\chi_+>=\varepsilon_i<\underset{i}{e}^{(-\alpha)}_r,\underset{i}{e}^{(+\beta)}_{r'}>=\delta_{rr'}\delta_{\alpha\beta}. \tag{2.18}$$

Taking into account two-valuedness of degenerate spin states it follows that the quanta are the fermions with half-integral spins. That is, the functions $\underset{\eta}{\Psi}^\lambda(\eta)$ and $\underset{u}{\Psi}^\lambda(u)$ may be regarded as the Fermi fields of $\eta$- and $u$-type quanta. Explicitly the matrix elements read

$$\begin{aligned} &\Phi^2_+(\zeta)=<\chi_+\mid\underset{\eta}{\hat{\Psi}}(\eta)\,\underset{\eta}{\hat{\bar{\Psi}}}(\eta)+\underset{u}{\hat{\Psi}}(u)\,\underset{u}{\hat{\bar{\Psi}}}(u)\mid\chi_+>=\varepsilon_i\underset{i}{\Psi}^{(+\alpha)}\underset{i}{\Psi}_{(+\alpha)}= \\ &=\underset{\eta}{\Psi}^{(+\alpha)}(\eta)\underset{\eta}{\bar{\Psi}}^{(+\alpha)}(\eta)-\underset{u}{\Psi}^{(+\alpha)}(u)\underset{u}{\bar{\Psi}}^{(+\alpha)}(u), \\ &\Phi^2_-(\zeta)=<\chi_-\mid\underset{\eta}{\hat{\Psi}}(\eta)\,\underset{\eta}{\hat{\bar{\Psi}}}(\eta)+\underset{u}{\hat{\Psi}}(u)\,\underset{u}{\hat{\bar{\Psi}}}(u)\mid\chi_->=\varepsilon_i\underset{i}{\Psi}^{(-\alpha)}\underset{i}{\Psi}_{(-\alpha)}= \\ &=\underset{\eta}{\Psi}^{(-\alpha)}(\eta)\underset{\eta}{\bar{\Psi}}^{(-\alpha)}(\eta)-\underset{u}{\Psi}^{(-\alpha)}(u)\underset{u}{\bar{\Psi}}^{(-\alpha)}(u). \end{aligned} \tag{2.19}$$

# 3 Operator Manifold $\hat{G}(2.2.3)$

The field aspect will be the subject for discussion in this section. Our notation will be that of the textbook by [7]. As far as a quantum may be regarded as a fermion field, its description is provided by the theory, which is in close analogy to Dirac's conventional wave-mechanical theory of fermions with spin $\frac{\vec{1}}{2}$ treated in terms of manifold $G(2.2.3)$ [1]. The final formulation of quantum theory is equivalent to configuration space wave mechanics with antisymmetric state functions. To facilitate our approach it seems useful to present some formal matters which one will have to know in order to understand the structure of theory. Below we proceed with preliminary discussion. The quantum field may be considered as bi-spinor field $\Psi(\zeta)$ defined on manifold $G(2.2.3) = \underset{\eta}{G}(2.3) \oplus \underset{u}{G}(2.3)$: $\Psi(\zeta) = \underset{\eta}{\Psi}(\eta)\,\underset{u}{\Psi}(u)$, where the $\underset{i}{\Psi}$ is a bi-spinor defined on the manifold $\underset{i}{G}(2.3)$. The state of free quantum of $i$-type with definite values of link-momentum $\underset{i}{p}$ and spin projection $s$ is described by means of plane waves, respectively (in units $\hbar = 1, \quad c = 1$):

$$\underset{\eta\, p_\eta}{\Psi}(\eta) = \left(\frac{m}{E_\eta}\right)^{1/2} \underset{\eta}{u}(p_\eta, s) e^{-ip_\eta \eta}, \quad \underset{u\, p_u}{\Psi}(u) = \left(\frac{m}{E_u}\right)^{1/2} \underset{u}{u}(p_u, s) e^{-ip_u u}, \tag{3.1}$$

where $E_i \equiv \underset{i\,0}{p} = (\underset{i\,0\alpha}{p}, \underset{i\,0\alpha}{p})^{1/2}, \quad \underset{i\,0\alpha}{p} = \frac{1}{\sqrt{2}}(\underset{i\,(+\alpha)}{p} + \underset{i\,(-\alpha)}{p})$. It is necessary to consider also the solutions of negative frequencies

$$\underset{\eta\, -p_\eta}{\Psi}(\eta) = \left(\frac{m}{E_\eta}\right)^{1/2} \underset{\eta}{\nu}(p_\eta, s) e^{ip_\eta \eta}, \quad \underset{u\, -p_u}{\Psi}(u) = \left(\frac{m}{E_u}\right)^{1/2} \underset{u}{\nu}(p_u, s) e^{ip_u u}, \tag{3.2}$$

where $\underset{i\,\alpha}{p} = \frac{1}{\sqrt{2}}(\underset{i\,(+\alpha)}{p} - \underset{i\,(-\alpha)}{p}), \quad p_\eta^2 = E_\eta^2 - \vec{p}_\eta^{\,2} = p_u^2 = E_u^2 - \vec{p}_u^{\,2} = m^2$. For the spinors the useful relations of orthogonality and completeness hold. We make use of localized wave packets constructed by means of superposition of plane wave solutions furnished by creation and annihilation operators in agreement with Pauli's principle

$$\underset{\eta}{\hat{\Psi}}(\eta) = \sum_{\pm s} \int \frac{d^3 p_\eta}{(2\pi)^{3/2}} \underset{\eta}{\hat{\Psi}}(p_\eta, s, \eta), \quad \underset{u}{\hat{\Psi}}(u) = \sum_{\pm s} \int \frac{d^3 p_u}{(2\pi)^{3/2}} \underset{u}{\hat{\Psi}}(p_u, s, u), \tag{3.3}$$

where, as usual, it is denoted

$$\begin{array}{ll} \underset{\eta}{\hat{\Psi}}(p_\eta, s, \eta) = \underset{\eta\,(\lambda\alpha)}{\hat{\gamma}}(p_\eta, s) \underset{\eta}{\Psi}^{(\lambda\alpha)}(p_\eta, s, \eta), & \underset{\eta}{\hat{\bar{\Psi}}}(p_\eta, s, \eta) = \underset{\eta}{\hat{\gamma}}^{(\lambda\alpha)}(p_\eta, s) \underset{\eta\,(\lambda\alpha)}{\Psi}(p_\eta, s, \eta), \\ \underset{u}{\hat{\Psi}}(p_u, s, u) = \underset{u\,(\lambda\alpha)}{\hat{\gamma}}(p_u, s) \underset{u}{\Psi}^{(\lambda\alpha)}(p_u, s, u), & \underset{u}{\hat{\bar{\Psi}}}(p_u, s, u) = \underset{u}{\hat{\gamma}}^{(\lambda\alpha)}(p_u, s) \underset{u\,(\lambda\alpha)}{\Psi}(p_u, s, u). \end{array} \tag{3.4}$$

One has $\underset{\eta}{\Psi}^{(\pm\alpha)} = \eta^{(\pm\alpha)} \underset{\eta}{\Psi}^{\pm}, \quad \underset{u}{\Psi}^{(\pm\alpha)} = u^{(\pm\alpha)} \underset{u}{\Psi}^{\pm}, \quad \underset{\eta\,(\pm\alpha)}{\Psi} = \eta_{(\pm\alpha)} \underset{\eta\,\pm}{\Psi}, \quad \underset{u\,(\pm\alpha)}{\Psi} = u_{(\pm\alpha)} \underset{u\,\pm}{\Psi}$, provided $\underset{i}{\Psi}^{+} \equiv \underset{i\,p_i}{\Psi}, \quad \underset{i}{\Psi}^{-} \equiv \underset{i\,-p_i}{\Psi}, \quad \underset{i\,\lambda}{\Psi} = \underset{i}{\bar{\Psi}}^{\lambda}, \quad \underset{i\,(\lambda\alpha)}{\Psi} = \underset{i}{\bar{\Psi}}^{(\lambda\alpha)}$. A closer examination of the properties of the matrix elements of the anticommutators of expansion coefficients

shows that

$$
\begin{array}{l}
<\chi_- \mid \{\hat{\underset{i}{\gamma}}^{(+\alpha)}(p_i,s), \hat{\underset{j}{\gamma}}_{(+\beta)}(p'_j,s')\} \mid \chi_- >= \\
=<\underset{i}{\hat{e}}^{(+\alpha)}(p_i,s), \underset{j}{\hat{e}}_{(+\beta)}(p'_j,s') >= \varepsilon_i \delta_{ij}\delta_{ss'}\delta_{\alpha\beta}\delta^{(3)}(\vec{p}_i - \vec{p'}_i), \\
<\chi_+ \mid \{\hat{\underset{j}{\gamma}}^{(-\beta)}(p'_j,s'), \hat{\underset{i}{\gamma}}_{(-\alpha)}(p_i,s)\} \mid \chi_+ >= \\
=<\underset{j}{\hat{e}}^{(-\beta)}(p'_j,s'), \underset{i}{\hat{e}}_{(-\alpha)}(p_i,s) >= \varepsilon_i \delta_{ij}\delta_{ss'}\delta_{\alpha\beta}\delta^{(3)}(\vec{p}_i - \vec{p'}_i).
\end{array}
\tag{3.5}
$$

We may also consider analogical wave packets of operator-vector fields $\hat{\Phi}(\zeta)$ and $\hat{\bar{\Phi}}(\zeta)$. While, explicitly the matrix element of anticommutator reads

$$
\begin{array}{l}
<\chi_\pm \mid \{\hat{\gamma}^{(\lambda,\mu,\alpha)}(p,s), \hat{\gamma}_{(\tau,\nu,\beta)}(p',s')\} \mid \chi_\pm >= \\
=< e^{(\lambda,\mu,\alpha)}(p,s), e_{(\tau,\nu,\beta)}(p',s') >= \delta_{ss'}\delta^\lambda_\tau \delta^\nu_\mu \delta^\alpha_\beta \delta^{(3)}(\vec{p}-\vec{p'}).
\end{array}
\tag{3.6}
$$

Thus

$$
\begin{array}{l}
\sum\limits_{\lambda=\pm} <\chi_\lambda \mid \hat{\Phi}(\zeta)\hat{\bar{\Phi}}(\zeta) \mid \chi_\lambda >= \sum\limits_{\lambda=\pm} <\chi_\lambda \mid \hat{\bar{\Phi}}(\zeta)\hat{\Phi}(\zeta) \mid \chi_\lambda >= \\
-i \lim\limits_{\zeta\to\zeta'} (\zeta\zeta') \underset{\zeta}{G}(\zeta-\zeta') = -i \left[ \lim\limits_{\eta\to\eta'} (\eta\eta') \underset{\eta}{G}(\eta-\eta') - \lim\limits_{u\to u'} (uu') \underset{u}{G}(u-u') \right],
\end{array}
\tag{3.7}
$$

where the Green's functions are used

$$
\underset{\eta}{G}(\eta-\eta') = -(i \underset{\eta}{\hat{\partial}} + m) \underset{\eta}{\Delta}(\eta-\eta'), \quad \underset{u}{G}(u-u') = -(i \underset{u}{\hat{\partial}} + m) \underset{u}{\Delta}(u-u'),
\tag{3.8}
$$

provided with the invariant singular functions $\underset{\eta}{\Delta}(\eta-\eta')$ and $\underset{u}{\Delta}(u-u')$. Meanwhile a second trend emerged as the quantum theory situation corresponding to simultaneously presence of many identical quanta. To describe the $n$-quanta fermion system by means of quantum field theory, it will be advantageous to make use of convenient method of constructing the state vector of physical system by proceeding from the vacuum state as a very point of origin. Exploiting the whole advantage of it, a particular emphasis will be placed just on the fact that, certainly, if there are $n$ identical quanta with coordinates $\zeta_1, \zeta_2, ..., \zeta_n$ the antisymmetrical state function $\Phi$ will be a function of all of them and presents the system of $n$ fermions $\Phi(\zeta_1, \zeta_2, ..., \zeta_n)$, which implies the Fermi-Dirac statistics.
It was assumed that the $i$-th quantum is found in the state $r_i$ with the field function $\Phi^{(\lambda_i,\mu_i,\alpha_i)}_{r_i} = \zeta^{(\lambda_i,\mu_i,\alpha_i)}_{r_i} \Phi^{\lambda_i,\mu_i}_{r_i}(\zeta_{r_i})$ and made use of following notation: $\zeta_{r_i^{\lambda_i,\mu_i}} = \sum\limits_{\alpha_i=1}^{3} e^{r_i}_{(\lambda_i,\mu_i,\alpha_i)} \zeta^{(\lambda_i,\mu_i,\alpha_i)}_{r_i}$; $\zeta_{r_i} = \sum\limits_{\lambda_i,\mu_i=1}^{2} \zeta_{r_i^{\lambda_i,\mu_i}} \in \widetilde{\mathcal{U}}_{r_i}$, the $\widetilde{\mathcal{U}}_{r_i}$ is the open neighborhood of the point $\zeta_{r_i}$; the $r_i$ implies a set $(r_i^{11}, r_i^{12}, r_i^{21}, r_i^{22})$. Let the $\mathcal{H}^{(1)}$ is a Hilbert space used for quantum mechanical description of one particle, namely $\mathcal{H}^{(1)}$ is a finite or infinite dimensional complex space, provided with scalar product $(\Phi, \Psi)$ being linear with respect to $\Psi$ and antilinear to $\Phi$. The $\mathcal{H}^{(1)}$ is complete in sense of norm $|\Phi| = (\Phi, \Phi)^{1/2}$, i.e. each fundamental sequence $\{\Phi_n\}$ of vectors of $\mathcal{H}^{(1)}$ converged by norm in $\mathcal{H}^{(1)}$. One-particle state function is written $\Phi^{(1)}_{r_i} = \prod\limits_{\lambda_i,\mu_i=1}^{2} \Phi^{(1)}_{r_i^{\lambda_i,\mu_i}} \in \mathcal{H}^{(1)}_{r_i}$, where $\mathcal{H}^{(1)}_{r_i} = \prod\limits_{\lambda_i,\mu_i=1}^{2} \otimes \mathcal{H}^{(1)}_{r_i^{\lambda_i,\mu_i}}$. So define

$$
\widetilde{\Phi}^{(1)} = \zeta_i \Phi^{(1)}_{r_i} \in \widetilde{G}^{(1)}_{r_i} = \widetilde{\mathcal{U}}^{(1)}_{r_i} \otimes \mathcal{H}^{(1)}_{r_i}.
\tag{3.9}
$$

For description of n-particle system we introduce Hilbert space

$$\bar{\mathcal{H}}^{(n)}_{(r_1,\ldots,r_n)} = \mathcal{H}^{(1)}_{r_1} \otimes \cdots \otimes \mathcal{H}^{(1)}_{r_n} \tag{3.10}$$

by considering all sequences

$$\Phi^{(n)}_{(r_1,\ldots,r_n)} = \{\Phi^{(1)}_{r_1}, \ldots, \Phi^{(1)}_{r_n}\} = \Phi^{(1)}_{r_1} \otimes \cdots \otimes \Phi^{(1)}_{r_n}, \tag{3.11}$$

where $\Phi^{(1)}_{r_i} \in \mathcal{H}^{(1)}_{r_i}$, provided, as usual, with the scalar product

$$(\Phi^{(n)}_{(r_1,\ldots,r_n)}, \Psi^{(n)}_{(r_1,\ldots,r_n)}) = \prod_{i=1}^{n} (\Phi^{(1)}_{r_i}, \Psi^{(1)}_{r_i}). \tag{3.12}$$

Obtaining the space eq.(3.9) we consider the space $\mathcal{H}^{(n)}_{(r_1,\ldots,r_n)}$ of all limited linear combinations of eq.(3.10) and continue by linearity the scalar product eq.(3.12) on $\mathcal{H}^{(n)}_{(r_1,ldots,r_n)}$. The wave function $\Phi^{(n)}_{(r_1,\ldots,r_n)} \in \mathcal{H}^{(n)}_{(r_1,\ldots,r_n)}$ must be antisymmetrized over its arguments. So, we ought to distinguish the antisymmetric part ${}^A\bar{\mathcal{H}}^{(n)}$ of Hilbert space $\bar{\mathcal{H}}^{(n)}$ by considering the functions

$${}^A\Phi^{(n)}_{(r_1,\ldots,r_n)} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma) \Phi^{(n)}_{\sigma(r_1,\ldots,r_n)}. \tag{3.13}$$

The summation is extended over all permutations of indices $(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu})$ of the integers $1, 2, \ldots, n$, whereas the antisymmetrical eigen-functions are sums of the same terms with alternating signs in dependence of a parity $sgn(\sigma)$ of transposition. In the aftermath, one continues by linearity on $\mathcal{H}^{(n)}$ the reflection $\Phi^{(n)} \to {}^A\Phi^{(n)}$ , which is limited and allowed the expansion by linearity on ${}^A\bar{\mathcal{H}}^{(n)}$. According to conventional definition, the region of values of this reflection is a ${}^A\bar{\mathcal{H}}^{(n)}$, namely the antisymmetrized tensor product of $n$ identical samples of $\mathcal{H}^{(1)}$. We introduce

$$\begin{gathered} {}^A\widetilde{\Phi}^{(n)}_{(r_1,\ldots,r_n)} = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma) \widetilde{\Phi}^{(n)}_{\sigma(r_1,\ldots,r_n)} = \\ = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S(n)} sgn(\sigma) \widetilde{\Phi}^{(1)}_{r_1} \otimes \cdots \otimes \widetilde{\Phi}^{(1)}_{r_n} \in {}^A\widetilde{G}^{(n)}_{(r_1,\ldots,r_n)} = \widetilde{\mathcal{U}}^{(n)}_{(r_1,\ldots,r_n)} \otimes {}^A\hat{\mathcal{H}}^{(n)}_{(r_1,\ldots,r_n)}. \end{gathered} \tag{3.14}$$

and constructing 12-dimensional wave manifold $\widetilde{G}(2.2.3)$ consider a set ${}^A\widetilde{\mathcal{F}}$ of all sequences

$${}^A\widetilde{\Phi} = \{{}^A\widetilde{\Phi}^{(0)}, {}^A\widetilde{\Phi}^{(1)} \ldots, {}^A\widetilde{\Phi}^{(n)} \ldots\}, \tag{3.15}$$

with a finite number of non-zero elements. Therewith, the set ${}^A\mathcal{F}$

$${}^A\Phi = \{{}^A\Phi^{(0)}, {}^A\Phi^{(1)} \ldots, {}^A\Phi^{(n)} \ldots\} \tag{3.16}$$

is provided with the structure of Hilbert sub-space by introducing the composition rules

$$\begin{gathered} {}^A(\lambda\Phi + \mu\Psi)^{(n)} = \lambda {}^A\Phi^{(n)} + \mu {}^A\Psi^{(n)}, \quad \forall \lambda, \mu \in C, \\ ({}^A\Phi, {}^A\Psi) = \sum_{n=0}^{\infty} ({}^A\Phi^{(n)}, {}^A\Psi^{(n)}). \end{gathered} \tag{3.17}$$

In the sequel, the wave manifold $\widetilde{G}(2.2.3)$ stems from the ${}^{A}\widetilde{\mathcal{F}}$ by making expansion by metric induced as a scalar product in ${}^{A}\mathcal{F}$

$$\widetilde{G}(2.2.3) = \sum_{n=0}^{\infty} \widetilde{G}^{(n)} = \sum_{n=0}^{\infty} \left( \widetilde{\mathcal{U}}^{(n)} \otimes {}^{A}\bar{\mathcal{H}}^{(n)} \right). \tag{3.18}$$

In general, the creation $\hat{\gamma}_{r}^{(\lambda,\mu,\alpha)}$ and annihilation $\hat{\gamma}^{r}_{(\lambda,\mu,\alpha)}$ operators for each $\mathcal{H}^{(1)}$ can be defined as follows. One ought to modify the operators eq.(2.10) in order to provide an anticommutation being valid in both cases acting on the same as well as different states:

$$\hat{\gamma}^{r}_{(\lambda,\mu,\alpha)} \Rightarrow \hat{\gamma}^{r}_{(\lambda,\mu,\alpha)}\eta_{r}^{\lambda\mu}, \quad \hat{\gamma}_{r}^{(\lambda,\mu,\alpha)} \Rightarrow \eta_{r}^{\lambda\mu}\hat{\gamma}_{r}^{(\lambda,\mu,\alpha)} = \hat{\gamma}_{r}^{(\lambda,\mu,\alpha)}\eta_{r}^{\lambda\mu}, \quad \left(\eta_{r}^{\lambda\mu}\right)^{+} = \eta_{r}^{\lambda\mu}, \tag{3.19}$$

for fixed $\lambda, \mu, \alpha$, where $\eta_{r}^{\lambda\mu}$ is a diagonal operator in the space of occupation numbers. Therewith, at $r_i < r_j$ one gets

$$\hat{\gamma}^{r_i}_{(\lambda,\mu,\alpha)}\eta_{r_j}^{\lambda\mu} = -\eta_{r_j}^{\lambda\mu}\hat{\gamma}^{r_i}_{(\lambda,\mu,\alpha)}, \quad \hat{\gamma}^{r_j}_{(\lambda,\mu,\alpha)}\eta_{r_i}^{\lambda\mu} = \eta_{r_i}^{\lambda\mu}\hat{\gamma}^{r_j}_{(\lambda,\mu,\alpha)}. \tag{3.20}$$

The operators of corresponding occupation numbers, for fixed $\lambda, \mu, \alpha$, are

$$\hat{N}_{r}^{\lambda\mu} = \hat{\gamma}^{r}_{(\lambda,\mu,\alpha)}\hat{\gamma}_{r}^{(\lambda,\mu,\alpha)} = \hat{N}_{r}^{\lambda} \otimes \hat{N}_{r}^{\mu}, \tag{3.21}$$

where we make use of (see eq.(2.5))

$$\hat{N}_{r}^{\lambda} = \begin{cases} \hat{O}_{1}^{r}\hat{O}_{r}^{1} = (\alpha_1\alpha_2)_r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_r, \\ \\ \hat{O}_{2}^{r}\hat{O}_{r}^{2} = (\alpha_2\alpha_1)_r = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_r. \end{cases} \tag{3.22}$$

As far as diagonal operators $(1 - 2\hat{N}_{r}^{\lambda\mu})$ anticommute with the $\hat{\gamma}^{r}_{(\lambda,\mu,\alpha)}$, then

$$\eta_{r_i}^{\lambda\mu} = \prod_{r=1}^{r_i-1}(1 - 2\hat{N}_{r}^{\lambda\mu}). \tag{3.23}$$

Combining eq.(3.20)-eq.(3.23), the explicit forms read

$$\begin{gathered} \eta_{r_i}^{11} = \prod_{r=1}^{r_i-1} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_r, \quad \eta_{r_i}^{21} = \prod_{r=1}^{r_i-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_r, \\ \eta_{r_i}^{12} = \prod_{r=1}^{r_i-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_r, \quad \eta_{r_i}^{22} = \prod_{r=1}^{r_i-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_r, \end{gathered} \tag{3.24}$$

provided

$$
\begin{aligned}
&\eta^{11}_{r_i}\Phi(n_1,\ldots,n_N;0;0;0)=\prod_{r=1}^{r_i-1}(-1)^{n_r}\Phi(n_1,\ldots,n_N;0;0;0),\\
&\eta^{12}_{r_i}\Phi(0;m_1,\ldots,n_M;0;0)=\prod_{r=1}^{r_i-1}(-1)^{m_r}\Phi(0;m_1,\ldots,m_M;0;0),\\
&\eta^{21}_{r_i}\Phi(0;0;q_1,\ldots,q_Q;0)=\prod_{r=1}^{r_i-1}(-1)^{q_r}\Phi(0;0;q_1,\ldots,q_Q;0),\\
&\eta^{22}_{r_i}\Phi(0;0;0;t_1,\ldots,t_T)=\prod_{r=1}^{r_i-1}(-1)^{t_r}\Phi(0;0;0;t_1,\ldots,t_T),
\end{aligned}
\tag{3.25}
$$

Here the occupation numbers $n_r(m_r,q_r,t_r)$ are introduced, which refer to the $r$-th states corresponding to operators $\hat{\gamma}^r_{(1,1,\alpha)}(\hat{\gamma}^r_{(1,2,\alpha)},\hat{\gamma}^r_{(2,1,\alpha)},\hat{\gamma}^r_{(2,2,\alpha)})$ either empty $(n_r,\ldots,t_r=0)$ or occupied $(n_r,\ldots,t_r=1)$. To save writing we abbreviate the modified operators by the same symbols. For example, the creation operator $\hat{\gamma}^{(\lambda,\mu,\alpha)}_{r_i}$ by acting on free state $\mid 0>_{r_i}$ yields the one-occupied state $\mid 1>_{r_i}$ with the phase $+$ or $-$ depending of parity of the number of quanta in the states $r<r_i$. Modified operators satisfy the same anticommutation relations of the operators eq.(2.10). It is convenient to make use of notation $\hat{\gamma}^{(\lambda,\mu,\alpha)}_r\equiv e^{(\lambda,\mu,\alpha)}_r\hat{b}^{\lambda\mu}_{(r\alpha)}$, $\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\equiv e^r_{(\lambda,\mu,\alpha)}\hat{b}^{(r\alpha)}_{\lambda\mu}$, and abbreviate the pair of indices $(r\alpha)$ by the single symbol $r$. Then, for each $\Phi\in {}^A\mathcal{H}^{(n)}$

$$
\Phi=\frac{1}{\sqrt{n!}}\sum_{\sigma\in S(n)}sgn(\sigma)\Phi^{(n)}_{\sigma(r_1,\ldots,r_n)},
\tag{3.26}
$$

where $\Phi^{(n)}_{\sigma(r_1,\ldots,r_n)}=\Phi^{(1)}_{\sigma(1)}\otimes\cdots\otimes\Phi^{(1)}_{\sigma(n)}$, and for any vector $f\in\mathcal{H}^{(1)}$, the operators $\hat{b}(f)$ $(\hat{b}^r_{\lambda\mu}(f))$ and $\hat{b}^*(f)$ $(\hat{b}^{\lambda\mu}_r(f))$ imply

$$
\begin{aligned}
\hat{b}(f)\Phi&=\frac{1}{\sqrt{(n-1)!}}\sum_{\sigma\in S(n)}sgn(\sigma)\left(f\,\Phi^{(1)}_{\sigma(1)}\right)\Phi^{(1)}_{\sigma(2)}\otimes\cdots\otimes\Phi^{(1)}_{\sigma(n)},\\
\hat{b}^*(f)\Phi&=\frac{1}{\sqrt{(n+1)!}}\sum_{\sigma\in S(n+1)}sgn(\sigma)\Phi^{(1)}_{\sigma(0)}\otimes\Phi^{(1)}_{\sigma(1)}\otimes\cdots\otimes\Phi^{(1)}_{\sigma(n)},
\end{aligned}
\tag{3.27}
$$

where $\Phi^{(1)}_{(0)}\equiv f$. In the aftermath, one continues the $\hat{b}(f)$ and $\hat{b}^*(f)$ by linearity to linear reflections, which will be denoted by the same symbols, acting from ${}^A\mathcal{H}^{(n)}$ into ${}^A\mathcal{H}^{(n-1)}$ or ${}^A\mathcal{H}^{(n+1)}$, respectively. They were limited by the values $\sqrt{n}|f|$ and $\sqrt{(n+1)}|f|$. So, they may be expanded by continuation up to the reflections acting from ${}^A\bar{\mathcal{H}}^{(n)}$ into ${}^A\bar{\mathcal{H}}^{(n-1)}$ or ${}^A\bar{\mathcal{H}}^{(n+1)}$. Finally, they ought to be continued by linearity up to the linear operators acting from ${}^A\mathcal{F}$ into ${}^A\mathcal{F}$. They are defined on the same close region in ${}^A\bar{\mathcal{H}}^{(n)}$, namely in ${}^A\mathcal{F}$, which is invariant with respect to reflections $\hat{b}(f)$ and $\hat{b}^*(f)$. Hence, at $f_i,g_i\in\mathcal{H}^{(1)}$ $(i=1,\ldots,n;j=1,\ldots,m)$ all polynomials over $\{\hat{b}^*(f_i)\}$ and $\{\hat{b}(g_j)\}$ are completely defined on ${}^A\mathcal{F}$. While

$$
\begin{aligned}
&<1,1\mid\{\hat{b}^{11}_r(f),\hat{b}^{r'}_{11}(g)\}\mid 1,1>=<1,2\mid\{\hat{b}^{12}_r(f),\hat{b}^{r'}_{12}(g)\}\mid 1,2>=\\
&=<2,1\mid\{\hat{b}^{21}_r(f),\hat{b}^{r'}_{(21}(g)\}\mid 2,1>=<2,2\mid\{\hat{b}^{22}_r(f),\hat{b}^{r'}_{22}(g)\}\mid 2,2>=\delta^{r'}_r.
\end{aligned}
\tag{3.28}
$$

The mean values $< \varphi; \hat{b}_r^{\lambda\mu}(f)\hat{b}^r_{\lambda\mu}(f) >$ calculated at given $\lambda, \mu$ for any element $\Phi \in {}^A\mathcal{F}$ equal to mean values of the symmetric operator of occupation number in terms of $\hat{N}^r_{\lambda\mu} = \hat{b}_r^{\lambda\mu}(f)\hat{b}^r_{\lambda\mu}(f)$, with a wave function $f$ in the state describing by $\Phi$. Here, as usual, it was denoted $< \varphi; A\Phi >= Tr\, P_\varphi A = (\Phi, A\Phi)$ for each vector $\Phi \in \mathcal{H}$ with $|\Phi| = 1$, while the $P_\varphi$ is projecting operator onto one- dimensional space $\{\lambda\Phi \mid \lambda \in C\}$ generated by $\Phi$. Therewith, the probability of transition $\varphi \to \psi$ is given $Pr\{\varphi \mid \psi\} = |(\psi, \varphi)|^2$. Also, it was assumed that the linear operator $A$ is defined on the elements of linear manifold $\mathcal{D}(A)$ of $\mathcal{H}$ taking the values in $\mathcal{H}$. The $\mathcal{D}(A)$ is an everywhere closed region of definition of $A$, namely the closure of $\mathcal{D}(A)$ over the norm given in $\mathcal{H}$ coincides with $\mathcal{H}$. Meanwhile, the $\mathcal{D}(A)$ was included in $\mathcal{D}(A^*)$ and $A$ coincides with the reduction of $A^*$ on $\mathcal{D}(A)$, because $\mathcal{D}(A)$ is a symmetric operator, where the linear operator $A^*$ is maximal conjugated to $A$. That is, any operator $A'$ conjugated to $A$ - $(\Psi, A'\Phi) = (A'\Psi, \Phi)$ at all $\Phi \in \mathcal{D}(A)$ and $\Psi \in \mathcal{D}(A')$ coincides with the reduction of $A^*$ on some linear manifold $\mathcal{D}(A')$ included in $\mathcal{D}(A^*)$. Thus, the operator $A^{**}$ is closed symmetric expansion of operator $A$, namely it is a closure of $A$. Self-conjugated operator $A$ (the closure of which self-conjugated) allows only one self-conjugated expansion $A^{**}$. Thus, self-conjugated closure $\hat{N}$ of operator $\sum_{i=1}^{\infty} \hat{b}^*(f_i)\hat{b}(f_i)$, where $\{f_i \mid i = 1, \ldots, n\}$ is an arbitrary orthogonal basis in $\mathcal{H}^{(1)}$, was regarded as the operator of occupation number. For the vector $\chi^0 \in {}^A\mathcal{F}$ and $\chi^{0(n)} = \delta_{0n}$ one gets $< \chi^{0(n)}, \hat{N}(f) >= 0$ for all $f \in \mathcal{H}^{(1)}$. So, the $\chi^0$ is a vector of vacuum state without any particle: $\hat{b}(f)\chi^0 = 0$ for all $f \in \mathcal{H}^{(1)}$. If $f = \{f_i \mid i = 1, 2, \ldots\}$ is an arbitrary orthogonal basis in $\mathcal{H}^{(1)}$, then due to irreducibility of operators $\hat{b}^*(f_i) \mid f_i \in f$, the ${}^A\mathcal{H}$ includes the 0 and whole space ${}^A\mathcal{H}$ as invariant sub-spaces with respect to all $\hat{b}^*(f)$.
In the sequel, to construct the 12-dimensional operator-manifold $\hat{G}(2.2.3)$ we consider a set $\hat{\mathcal{F}}$ of all sequences $\hat{\Phi} = \{\hat{\Phi}^{(0)}, \hat{\Phi}^{(1)}, \ldots, \hat{\Phi}^{(n)}, \ldots\}$ with a finite number of non-zero elements, provided

$$\begin{aligned}
&\hat{\Phi}^{(n)}_{(r_1,\ldots,r_n)} = \hat{\Phi}^{(1)}_{r_1} \otimes \cdots \otimes \hat{\Phi}^{(1)}_{r_n} \in \hat{G}^{(n)}, \quad \hat{\Phi}^{(1)}_{r_i} = \hat{\zeta}_{r_i}\Phi^{(1)}_{r_i} \in \hat{G}^{(1)}_i = \hat{\mathcal{U}}^{(1)}_i \otimes \mathcal{H}^{(1)}_i, \\
&\hat{\zeta}_{r_i} \equiv \sum_{\alpha_i=1}^{3} \hat{\gamma}^{r_i}_{(\lambda_i,\mu_i,\alpha_i)}\zeta^{(\lambda_i,\mu_i,\alpha_i)}_{r_i} \in \hat{\mathcal{U}}^{(1)}_{r_i}, \quad \hat{G}^{(n)} = \hat{\mathcal{U}}^{(n)} \otimes \bar{\mathcal{H}}^{(n)}, \\
&\hat{\mathcal{U}}^{(n)}_{(r_1,\ldots,r_n)} = \hat{\mathcal{U}}^{(1)}_{r_1}, \otimes \cdots \otimes \hat{\mathcal{U}}^{(1)}_{r_n}.
\end{aligned} \tag{3.29}$$

Then, by analogy with eq.(3.18), the operator manifold $\hat{G}(2.2.3)$ ensued

$$\hat{G}(2.2.3) = \sum_{n=0}^{\infty} \hat{G}^{(n)} = \sum_{n=0}^{\infty} \left(\hat{\mathcal{U}}^{(n)} \otimes \bar{\mathcal{H}}^{(n)}\right), \tag{3.30}$$

which decomposed as follows: $\hat{G}(2.2.3) = {}^*\hat{\mathbf{R}}^{22} \otimes \hat{\mathbf{R}}^3$, where the linear unit operator bi-pseudo vectors $\{\hat{O}^{r_1r_2}_{\lambda,\mu} \equiv \hat{O}^{r_1}_{\lambda} \otimes \hat{O}^{r_2}_{\mu}\}$ served as the basis for operator-vectors of tangent sections of the $2 \times 2$-dimensional linear bi-pseudo operator-space ${}^*\hat{\mathbf{R}}^{22}$, and $\hat{\mathbf{R}}^3$ is the three-dimensional real linear operator-space with the basis for tangent operator-vectors consisting of the ordinary unit operator-vectors $\{\hat{\sigma}^r_\alpha\}$.
At last it is useful to bring a rigorous definition of secondary quantized form of one-particle observable $A$ in $\mathcal{H}$. Following to [8], let consider a set of identical samples $\hat{\mathcal{H}}_i$ of one-particle space $\mathcal{H}^{(1)}$ and operators $A_i$ acting in them. To each closed linear operator $A^{(1)}$ in $\mathcal{H}^{(1)}$ with the everywhere closed region of definition $\mathcal{D}(A^{(1)})$ following operators

correspond

$$
\begin{array}{l}
A_1^{(n)} = A^{(1)} \otimes I \otimes \cdots \otimes I, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_n^{(n)} = I \otimes I \otimes \cdots \otimes A^{(1)}.
\end{array}
\tag{3.31}
$$

Their sum $\sum_{j=1}^{n} A_j^{(n)}$ is given on the intersection of regions of definition of operator-terms including a linear manifold $\mathcal{D}(A^{(1)}) \otimes \cdots \otimes \mathcal{D}(A^{(n)})$ being close in $\hat{\mathcal{H}}^{(n)}$. While, the $A^{(n)}$ is a minimal closed expansion of this sum with $\mathcal{D}(A^{(n)})$. One considered linear manifold $\mathcal{D}(\Omega(A))$ in $\mathcal{H} = \sum_{n=0}^{\infty} \hat{\mathcal{H}}^{(n)}$ defined as a set of all vectors $\Psi \in \mathcal{H}$ such as $\Psi^{(n)} \in \mathcal{D}(A^{(n)})$ and $\sum_{n=0}^{\infty} \left| A^{(n)}\Psi^{(n)} \right|^2 < \infty$ . The manifold $\mathcal{D}(\Omega(A))$ is closed in $\mathcal{H}$. On this manifold one defines a closed linear operator $\Omega(A)$ acting as $\Omega(A)^{(n)} = A^{(n)}\Psi^{(n)}$, that is $\Omega(A)\Phi = \sum_{n=0}^{\infty} A^{(n)}\Psi^{(n)}$, while the $\Omega(A)$ is self-conjugated operator with everywhere closed region of definition. We think of vector $\Phi^{(n)} \in \mathcal{H}^{(n)}$ being in the form eq.(3.11), where $\Phi_i \in \mathcal{D}(A)$, then $< \varphi^{(n)}; A^{(n)} >= \sum_{i=1}^{n} < \varphi_i; A >$, which allows an expansion by continuing onto $\mathcal{D}(A)$. Thus, the $A^{(n)}$ is $n$-particle observable corresponding to one-particle observable $A$. So. one gets $< \varphi; \Omega(A) >= \sum_{n=0}^{\infty} < \varphi^{(n)}; A^{(n)} >$ for any $\Phi_i \in \mathcal{D}(\Omega(A))$. While, the $\Omega(A)$ reflects ${}^A\mathcal{D} = \mathcal{D}(\Omega(A)) \frown {}^A\mathcal{H}$ into ${}^A\mathcal{H}$. The reduction of $\Omega(A)$ on ${}^A\mathcal{H}$ is self-conjugated in the region ${}^A\mathcal{D}$, because of the fact that ${}^A\mathcal{H}$ is closed sub-space of $\mathcal{H}$. Hence, the $\Omega(A)$ is a secondary quantized form of one-particle observable $A$ in $\mathcal{H}$.

The main features of quantum field aspect will be completed below by constructing the explicit forms of wave state functions and calculating the matrix elements of concrete field operators. The vacuum state reads eq.(2.16), with the normalization requirement

$$
< \chi^0(\nu_1', \nu_2', \nu_3', \nu_4') \mid \chi^0(\nu_1, \nu_2, \nu_3, \nu_4) >= \prod_{i=1}^{4} \delta_{\nu_i \nu_i'}. \tag{3.32}
$$

The state vectors

$$
\begin{array}{l}
\chi(\{n_r\}_1^N; \{m_r\}_1^M; \{q_r\}_1^Q; \{t_r\}_1^T; \{\nu_r\}_1^4) = (\hat{b}_N^{11})^{n_N} \cdots (\hat{b}_1^{11})^{n_1} \cdot \\
\cdot (\hat{b}_M^{12})^{m_M} \cdots (\hat{b}_1^{12})^{m_1} \cdot (\hat{b}_Q^{21})^{q_Q} \cdots (\hat{b}_1^{21})^{q_1} \cdot \cdot (\hat{b}_T^{22})^{t_T} \cdots (\hat{b}_1^{22})^{t_1} \chi^0(\nu_1, \nu_2, \nu_3, \nu_4),
\end{array}
\tag{3.33}
$$

where $\{n_r\}_1^N = n_1, \ldots, n_N$ and so on, are the eigen-functions of modified operators. They form a whole set of orthogonal vectors

$$
\begin{array}{l}
< \chi(\{n_r'\}_1^N; \{m_r'\}_1^M; \{q_r'\}_1^Q; \{t_r'\}_1^T; \{\nu_r'\}_1^4) \mid \chi(\{n_r\}_1^N; \{m_r\}_1^M; \{q_r\}_1^Q; \{t_r\}_1^T; \{\nu_r\}_1^4) >= \\
= \prod_{r=1}^{N} \delta_{n_r n_r'} \cdot \prod_{r=1}^{M} \delta_{m_r m_r'} \cdot \prod_{r=1}^{Q} \delta_{q_r q_r'} \cdot \prod_{r=1}^{T} \delta_{t_r t_r'} \cdot \prod_{r=1}^{4} \delta_{\nu_r \nu_r'}.
\end{array}
\tag{3.34}
$$

Considering an arbitrary superposition

$$\chi = \sum_{\substack{n_1,\ldots,n_N = 0 \\ m_1,\ldots,m_M=0 \\ q_1,\ldots,q_Q=0 \\ t_1,\ldots,t_T=0}}^{1} c'(\{n_r\}_1^N;\{m_r\}_1^M;\{q_r\}_1^Q;\{t_r\}_1^T)\chi(\{n_r\}_1^N;\{m_r\}_1^M;\{q_r\}_1^Q;\{t_r\}_1^T;\{\nu_r\}_1^4), \tag{3.35}$$

the coefficients $c'$ of expansion are the corresponding amplitudes of probabilities:

$$<\chi \mid \chi>= \sum_{\substack{n_1,\ldots,n_N=0 \\ m_1,\ldots,m_M=0 \\ q_1,\ldots,q_Q=0 \\ t_1,\ldots,t_T=0}}^{1} \left| c'(\{n_r\}_1^N;\{m_r\}_1^M;\{q_r\}_1^Q;\{t_r\}_1^T)\right|^2. \tag{3.36}$$

Taking into account eq.(3.28), the non-vanishing matrix elements of operators $\hat{b}^{11}_{r_k}$ and $\hat{b}^{r_k}_{11}$ read

$$\begin{aligned}
&<\chi(\{n'_r\}_1^N;0;0;0;1,0,0,0) \mid \hat{b}^{11}_{r_k}\chi(\{n_r\}_1^N;0;0;0;1,0,0,0)>= \\
&=<1,1 \mid \hat{b}^{r'_1}_{11}\cdots\hat{b}^{r'_n}_{11}\cdot\hat{b}^{11}_{r_k}\cdot\hat{b}^{11}_{r_n}\cdots\hat{b}^{11}_{r_1} \mid 1,1>= \\
&=\begin{cases} (-1)^{n'-k'} & \text{if } n_r = n'_r \text{ for } r\neq r_k \text{ and } n_{r_k}=0; n'_{r_k}=1, \\ 0 & \text{otherwise,} \end{cases} \\
&<\chi(\{n'_r\}_1^N;0;0;0;1,0,0,0) \mid \hat{b}^{r_k}_{11}\chi(\{n_r\}_1^N;0;0;0;1,0,0,0)>= \\
&=<1,1 \mid \hat{b}^{r'_1}_{11}\cdots\hat{b}^{r'_n}_{11}\cdot\hat{b}^{r_k}_{11}\cdot\hat{b}^{11}_{r_n}\cdots\hat{b}^{11}_{r_1} \mid 1,1>= \\
&=\begin{cases} (-1)^{n-k} & \text{if } n_r = n'_r \text{ for } r\neq r_k \text{ and } n'_{r_k}=0; n_{r_k}=1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{3.37}$$

where one denoted

$$n = \sum_{r=1}^{N} n_r, \quad n' = \sum_{r=1}^{N} n'_r, \tag{3.38}$$

the $r_k$ and $r'_k$ are $k$-th and $k'$-th terms of regulated sets of $\{r_1,\ldots,r_n\}$ $(r_1 < r_2 < \cdots < r_n)$ and $\{r'_1,\ldots,r'_n\}$ $(r'_1 < r'_2 < \cdots < r'_n)$, respectively. Continuing along this line we get a whole set of explicit forms of matrix elements of the rest of operators $\hat{b}^{\lambda\mu}_{r_k}$ and $\hat{b}^{r_k}_{\lambda\mu}$. Thus

$$\begin{aligned}
&\sum_{\{\nu_r\}=0}^{1} <\chi^0 \mid \hat{\Phi}(\zeta) \mid \chi>= \sum_{\{\nu_r\}=0}^{1} <\chi^0 \mid \hat{\gamma}^r_{(\lambda,\mu,\alpha)}\Phi_r^{(\lambda,\mu,\alpha)}(\zeta) \mid \chi>= \\
&= \sum_{r=1}^{N} c'_{n_r} e^{n_r}_{(1,1,\alpha)}\Phi^{(1,1,\alpha)}_{n_r} + \sum_{r=1}^{M} c'_{m_r} e^{m_r}_{(1,2,\alpha)}\Phi^{(1,2,\alpha)}_{m_r} + \sum_{r=1}^{Q} c'_{q_r} e^{q_r}_{(2,1,\alpha)}\Phi^{(2,1,\alpha)}_{q_r} + \sum_{r=1}^{T} c'_{t_r} e^{t_r}_{(2,2,\alpha)}\Phi^{(2,2,\alpha)}_{t_r},
\end{aligned} \tag{3.39}$$

provided

$$
\begin{array}{ll}
c'_{n_r} \equiv \delta_{1n_r} c'(0,\dots,n_r,\dots,0;0;0;0), & c'_{m_r} \equiv \delta_{1m_r} c'(0;0,\dots m_r,\dots,0;0;0),\\
c'_{q_r} \equiv \delta_{1q_r} c'(0;0;0,\dots q_r,\dots,0;0), & c'_{t_r} \equiv \delta_{1t_r} c'(0;0;0;0,\dots t_r,\dots,0).
\end{array}
\tag{3.40}
$$

Hereinafter we change the notation to

$$
\begin{array}{llll}
\bar{c}(r^{11}) = c'_{n_r}, & \bar{c}(r^{21}) = c'_{q_r}, & N_{11} = N, & N_{21} = Q,\\
\bar{c}(r^{12}) = c'_{m_r}, & \bar{c}(r^{22}) = c'_{t_r}, & N_{12} = M, & N_{22} = T,
\end{array}
\tag{3.41}
$$

and make use of

$$
\begin{array}{l}
F_{r^{\lambda\mu}} = \sum_\alpha e^{r^{\lambda\mu}}_{(\lambda,\mu,\alpha)} \Phi^{(\lambda,\mu,\alpha)}_{r^{\lambda\mu}}, \quad F^{r^{\lambda\mu}} = \sum_\alpha e^{(\lambda,\mu,\alpha)}_{r^{\lambda\mu}} \Phi^{r^{\lambda\mu}}_{(\lambda,\mu,\alpha)} = \bar{F}_{r^{\lambda\mu}},\\
\sum_{\{\nu_r\}=0}^{1} < \chi^0 \mid \hat{A} \mid \chi > \equiv < \chi^0 \parallel \hat{A} \parallel \chi >, \quad \sum_{\{\nu_r\}=0}^{1} < \chi \mid \hat{A} \mid \chi^0 > \equiv < \chi \parallel \hat{A} \parallel \chi^0 > .
\end{array}
\tag{3.42}
$$

Then, the matrix elements of operator-vector and co-vector fields take the forms

$$
\begin{array}{l}
< \chi^0 \parallel \hat{\Phi}(\zeta) \parallel \chi > = \sum_{\lambda\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r^{\lambda\mu}) F_{r^{\lambda\mu}}(\zeta),\\
< \chi \parallel \bar{\hat{\Phi}}(\zeta) \parallel \chi^0 > = \sum_{\lambda\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}^*(r^{\lambda\mu}) F^{r^{\lambda\mu}}(\zeta).
\end{array}
\tag{3.43}
$$

Explicitly the matrix element of $n$-operator vector fields reads

$$
\begin{array}{l}
\frac{1}{\sqrt{n!}} < \chi^0 \parallel \hat{\Phi}(\zeta_1) \cdots \hat{\Phi}(\zeta_n) \parallel \chi > =\\
= \frac{1}{\sqrt{n!}} \left\{ \sum_{\lambda\mu=1}^{2} \right\}_1^n \sum_{r_1^{\lambda\mu},\dots r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu},\dots,r_n^{\lambda\mu}) \sum_{\sigma\in S_n} sgn(\sigma) F_{r_1^{\lambda\mu}}(\zeta_1)\cdots F_{r_n^{\lambda\mu}}(\zeta_n) =\\
= \frac{1}{\sqrt{n!}} \left\{ \sum_{\lambda\mu=1}^{2} \right\}_1^n \sum_{r_1^{\lambda\mu},\dots r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu},\dots,r_n^{\lambda\mu}) \left\| \begin{array}{c} F_{r_1^{\lambda\mu}}(\zeta_1)\cdots F_{r_n^{\lambda\mu}}(\zeta_1)\\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ F_{r_1^{\lambda\mu}}(\zeta_n)\cdots F_{r_n^{\lambda\mu}}(\zeta_n) \end{array} \right\| .
\end{array}
\tag{3.44}
$$

where $\left\{ \sum_{\lambda\mu}^{2} \right\}_1^n \equiv \sum_{\lambda_1\mu_1}^{2} \cdots \sum_{\lambda_n\mu_n}^{2}$ and $r_i^{\lambda\mu} \equiv r^{\lambda_i\mu_i}$. Here we make use of

$$
\begin{array}{ll}
\bar{c}(r_1^{11},\dots,r_n^{11}) = c'(n_1,\dots,n_N;0;0;0), & \bar{c}(r_1^{12},\dots,r_n^{12}) = c'(0;m_1,\dots,m_M;0;0),\\
\bar{c}(r_1^{21},\dots,r_n^{21}) = c'(0;0;q_1,\dots,q_Q;0), & \bar{c}(r_1^{22},\dots,r_n^{22}) = c'(0;0;0;t_1,\dots,t_T).
\end{array}
\tag{3.45}
$$

Analogous expression can be obtained for co-vector fields too. Following to this procedure, the anticommutation relations ensued

$$
< \chi_- \mid \{ \hat{b}^{+}_{i_r}, \hat{b}^{r'}_{i_+} \} \mid \chi_- > = \delta_r^{r'}, \quad < \chi_+ \mid \{ \hat{b}^{-}_{i_r}, \hat{b}^{r'}_{i_-} \} \mid \chi_+ > = \delta_r^{r'},
\tag{3.46}
$$

provided as usual $\hat{\gamma}^{(\lambda\alpha)}_{i\,r} = \hat{e}^{(\lambda\alpha)}_{i\,r}\hat{b}^{\lambda}_{i\,(r\alpha)}$, $\quad \hat{\gamma}^{r}_{i\,(\lambda\alpha)} = \hat{e}^{r}_{i\,(\lambda\alpha)}\hat{b}^{(r\alpha)}_{i\,\lambda}$, $\quad (r\alpha) \Rightarrow r$, where the functions $\chi_{\pm}$ have the form eq.(2.16). The state functions

$$
\begin{aligned}
&\chi(\{n_r\}^N_1; \{m_r\}^M_1; \{q_r\}^Q_1; \{t_r\}^T_1; \lambda; \mu) = \\
&= (\hat{b}^{+}_{\eta_N})^{n_N} \cdots (\hat{b}^{+}_{\eta_1})^{n_1} \cdot (\hat{b}^{-}_{\eta_M})^{m_M} \cdots (\hat{b}^{-}_{\eta_1})^{m_1} \cdot (\hat{b}^{+}_{uQ})^{q_Q} \cdots (\hat{b}^{+}_{u1})^{q_1} \cdot \\
&\cdot (\hat{b}^{-}_{uT})^{t_T} \cdots (\hat{b}^{-}_{u1})^{t_1} \cdot \chi_-(\lambda)\chi_+(\mu),
\end{aligned}
\tag{3.47}
$$

are normalized eq.(3.32) and formed a whole set of orthogonal eigen-functions of the corresponding operators of occupation numbers $\hat{N}^{\lambda}_{i\,r} = \hat{b}^{\lambda}_{i\,r}\hat{b}^{r}_{i\,\lambda}$. They have the corresponding expectation values 0,1. Finally we introduce the function $\chi$, which is an arbitrary superposition and has the form eq.(3.35). Incorporating with anticommutation relations eq.(3.46) one gets a whole set of non-vanishing matrix elements of operators $\hat{b}^{\lambda}_{i\,r}$ and $\hat{b}^{r}_{i\,\lambda}$. So define

$$
\begin{aligned}
&\sum_{\lambda=1}^{2} < \chi(\{n'_r\}^N_1; 0; 0; 0; \lambda; 0) \mid \hat{b}^{+}_{\eta_{r_k}} \chi(\{n_r\}^N_1; 0; 0; 0; \lambda; 0) >= \\
&= \begin{cases} (-1)^{n'-k'} & \text{if } n_r = n'_r \text{ for } r \neq r_k \text{ and } n_{r_k} = 0; n'_{r_k} = 1, \\ 0 & \text{otherwise,} \end{cases} \\
&\sum_{\lambda=1}^{2} < \chi(\{n'_r\}^N_1; 0; 0; 0; \lambda; 0) \mid \hat{b}^{r_k}_{\eta_+} \chi(\{n_r\}^N_1; 0; 0; 0; \lambda; 0) >= \\
&= \begin{cases} (-1)^{n-k} & \text{if } n_r = n'_r \text{ for } r \neq r_k \text{ and } n'_{r_k} = 0; n_{r_k} = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
\tag{3.48}
$$

or

$$
\begin{aligned}
&\sum_{\mu=1}^{2} < \chi(0; \{m'_r\}^M_1; 0; 0; 0; \mu) \mid \hat{b}^{-}_{\eta_{r_k}} \chi(0; \{m_r\}^M_1; 0; 0; 0; \mu) >= \\
&= \begin{cases} (-1)^{m'-k'} & \text{if } m_r = m'_r \text{ for } r \neq r_k \text{ and } m_{r_k} = 0; m'_{r_k} = 1, \\ 0 & \text{otherwise,} \end{cases} \\
&\sum_{\mu=1}^{2} < \chi(0; \{m_r\}^N_1; 0; 0; 0; \mu) \mid \hat{b}^{r_k}_{\eta_-} \chi(0; \{m_r\}^M_1; 0; 0; 0; \mu) >= \\
&= \begin{cases} (-1)^{m-k} & \text{if } m_r = m'_r \text{ for } r \neq r_k \text{ and } m'_{r_k} = 0; m_{r_k} = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
\tag{3.49}
$$

where

$$
n = \sum_{r=1}^{N} n_r, \quad n' = \sum_{r=1}^{N} n'_r, \quad m = \sum_{r=1}^{M} m_r, \quad m' = \sum_{r=1}^{M} m'_r,
\tag{3.50}
$$

and so forth. In used notation we get

$$< \chi^0 \parallel \underset{\eta}{\hat{\Psi}}(\eta) \parallel \chi >= \sum_{\lambda=1}^{2}\sum_{r^\lambda}^{N_\lambda} \bar{c}(r^\lambda) \underset{\eta}{F}_{r^\lambda}(\eta), \quad < \chi \parallel \underset{\eta}{\bar{\hat{\Psi}}}(\eta) \parallel \chi^0 >= \sum_{\lambda=1}^{2}\sum_{r^\lambda=1}^{N_\lambda} \bar{c}^*(r^\lambda) \underset{\eta}{\bar{F}}_{r^\lambda}(\eta). \tag{3.51}$$

and

$$< \chi^0 \parallel \underset{u}{\hat{\Psi}}(u) \parallel \chi >= \sum_{\lambda=1}^{2}\sum_{r^\lambda=1}^{N_\lambda} \bar{c}(r^\lambda) \underset{u}{F}_{r^\lambda}(u), \quad < \chi \parallel \underset{u}{\bar{\hat{\Psi}}}(u) \parallel \chi^0 >= \sum_{\lambda=1}^{2}\sum_{r^\lambda=1}^{N_\lambda} \bar{c}^*(r^\lambda) \underset{u}{\bar{F}}_{r^\lambda}(u). \tag{3.52}$$

provided

$$\underset{\eta}{F}_{r^\lambda}(\eta) = \sum_\alpha \underset{\eta}{e}^{(\lambda\alpha)}_{r^\lambda} \underset{\eta}{\Psi}^{r^\lambda}_{(\lambda\alpha)}(\eta), \quad \underset{u}{F}_{r^\lambda}(u) = \sum_\alpha \underset{u}{e}^{(\lambda\alpha)}_{r^\lambda} \underset{u}{\Psi}^{r^\lambda}_{(\lambda\alpha)}(u), \tag{3.53}$$

$$N_1^1 = N, \quad N_2^1 = M, \quad N_1^2 = Q, \quad N_2^2 = T. \tag{3.54}$$

Thus

$$\begin{aligned} < \chi^0 \parallel \underset{\eta}{\hat{\Psi}}(\eta) \underset{\eta}{\bar{\hat{\Psi}}}(\eta') \parallel \chi^0 >= \sum_{\lambda\tau=1}^{2}\sum_{r^\lambda=1}^{N^1_\lambda}\sum_{r^\tau=1}^{N^1_\tau} \bar{c}(r^\lambda)\bar{c}^*(r^\tau) \underset{\eta}{F}_{r^\lambda}(\eta), \underset{\eta}{\bar{F}}_{r^\tau}(\eta'), \\ < \chi^0 \parallel \underset{u}{\hat{\Psi}}(u) \underset{u}{\bar{\hat{\Psi}}}(u') \parallel \chi^0 >= \sum_{\lambda\tau=1}^{2}\sum_{r^\lambda=1}^{N^2_\lambda}\sum_{r^\tau=1}^{N^2_\tau} \bar{c}(r^\lambda)\bar{c}^*(r^\tau) \underset{u}{F}_{r^\lambda}(u), \underset{u}{\bar{F}}_{r^\tau}(u'). \end{aligned} \tag{3.55}$$

# 4 Differential Geometric Aspect

The other interesting offshoot of this generalization is a geometric aspect, the major points of which will be discussed in this section. The set of operators $\{\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\}$ is the basis for all operator-vectors of tangent section $\hat{\mathbf{T}}_{\Phi_p}$ of principle bundle with the base $\hat{G}(2.2.3)$ at the point $\mathbf{\Phi}_p = \mathbf{\Phi}(\zeta(t))|_{t=0} \in \hat{G}(2.2.3)$. We assumed that the $\lambda(t) : R^1 \to G(2.2.3)$ is a maximal curve passing through the point $p = \lambda(0) \in \hat{G}(2.2.3)$. The smooth differentiable functions $\{\Phi^{(\lambda,\mu,\alpha)}(\zeta(t))\}$, belonging to the ring of functions of $C^\infty$-class, are local coordinates in the open neighborhood of the point $\mathbf{\Phi}_p \in \hat{\mathcal{U}}_\Phi$. Passing through $\mathbf{\Phi}_p$ a maximal curve has the coordinates $\{\Phi^{(\lambda,\mu,\alpha)}(\zeta(t))\}$. So, the smooth field of tangent operator-vector $\hat{\mathbf{A}}(\mathbf{\Phi}(\zeta))$ is a class of equivalency of the curves $\mathbf{f}(\mathbf{\Phi}(\zeta))$, $\mathbf{f}(\mathbf{\Phi}(\zeta(0))) = \mathbf{\Phi}_p$. While, the operator-differential $\hat{d}\, A^t_p$ of the flux $A^t_p : \hat{G}(2.2.3) \to \hat{G}(2.2.3)$ at the point $\mathbf{\Phi}_p$ with the velocity fields $\hat{\mathbf{A}}(\mathbf{\Phi}(\zeta))$ was defined by one-parameter group of diffeomorphismes given for the curve $\mathbf{\Phi}(\zeta(t)) : R^1 \to \hat{G}(2.2.3)$, provided $\mathbf{\Phi}(\zeta(0)) = \mathbf{\Phi}_p$ and $\hat{\dot{\mathbf{\Phi}}}(\zeta(0)) = \hat{\mathbf{A}}_p$

$$\hat{d}\, A^t_p(\mathbf{A}) = \left.\frac{\hat{d}}{d\,t}\right|_{t=0} A^t(\mathbf{\Phi}(\zeta(t))) = \hat{\mathbf{A}}(\mathbf{\Phi}(\zeta)) = \hat{\gamma}^r_{(\lambda,\mu,\alpha)} A^{(\lambda,\mu,\alpha)}_p, \tag{4.1}$$

where the $\{A^{(\lambda,\mu,\alpha)}_p\}$ are the components of $\hat{\mathbf{A}}$ in the basis $\{\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\}$. According to eq.(4.1), in holonomic coordinate basis $\hat{\gamma}^r_{(\lambda,\mu,\alpha)} \to \left(\hat{\partial} \,/\, \partial\Phi^{(\lambda,\mu,\alpha)}_r(\zeta(t))\right)_p$ one gets

$$A^{(\lambda,\mu,\alpha)}_p = \left.\frac{\partial\Phi^{(\lambda,\mu,\alpha)}_r}{\partial\zeta^{(\tau,\nu,\beta)}_r}\frac{d\,\zeta^{(\tau,\nu,\beta)}_r}{d\,t}\right|_p. \tag{4.2}$$

The operator-tensor $\hat{\mathbf{T}}$ of $\widehat{(n,0)}$-type at the point $\mathbf{\Phi}_p$ is a linear function belonging to the space

$$\hat{\mathbf{T}}^n_0 = \underbrace{\hat{\mathbf{T}}_{\Phi_p} \otimes \cdots \otimes \hat{\mathbf{T}}_{\Phi_p}}_{n}, \tag{4.3}$$

where $\otimes$ stands for tensor product. It sets up a correspondence between the element $(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n)$ of $\hat{\mathbf{T}}^n_0$ and the number $T(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n)$, provided by linearity

$$\begin{array}{l} T(\hat{\mathbf{A}}_1, \ldots, \alpha\hat{\mathbf{A}}_i, \ldots, \hat{\mathbf{A}}_n) = \alpha T(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n), \quad \forall\alpha \in R, \quad \hat{\mathbf{A}}_i \in \hat{\mathbf{T}}_{\Phi_p}, \\ (T_1 + T_2)(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n) = T_1(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n) + T_2(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n). \end{array} \tag{4.4}$$

In the basis $\{\hat{\gamma}^r_{(\lambda,\mu,\alpha)}\}$, for any $\hat{\mathbf{T}} \in \hat{\mathbf{T}}^n$, one has

$$\hat{\mathbf{T}} = T^{(\lambda_1,\mu_1,\alpha_1)\cdots(\lambda_n,\mu_n,\alpha_n)}_{r_1\cdots r_n} \hat{\gamma}^{r_1}_{(\lambda_1,\mu_1,\alpha_1)} \otimes \cdots \otimes \hat{\gamma}^{r_n}_{(\lambda_n,\mu_n,\alpha_n)} \tag{4.5}$$

To render our discussion here more transparent, below according to eq.(3.43), an explicit form of matrix element of operator-tensor $\hat{\mathbf{T}} = \hat{\Phi}(\zeta_1) \otimes \cdots \otimes \hat{\Phi}(\zeta_n)$ reads

$$\begin{array}{l} \dfrac{1}{\sqrt{n!}} < \chi^0 \parallel \hat{\Phi}(\zeta_1) \otimes \cdots \otimes \hat{\Phi}(\zeta_n) \parallel \chi >= \\ = \left\{ \displaystyle\sum_{\lambda\mu=1}^{2} \right\}^n_1 \displaystyle\sum_{r_1^{\lambda\mu},\ldots,r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) \sum_{\sigma\in S_n} sgn(\sigma) F_{r_1^{\lambda\mu}}(\zeta_1) \otimes \cdots \otimes F_{r_n^{\lambda\mu}}(\zeta_n) = \\ = \left\{ \displaystyle\sum_{\lambda\mu=1}^{2} \right\}^n_1 \displaystyle\sum_{r_1^{\lambda\mu},\ldots,r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) F_{r_1^{\lambda\mu}}(\zeta_1) \wedge \cdots \wedge F_{r_n^{\lambda\mu}}(\zeta_n), \end{array} \tag{4.6}$$

where $\wedge$ stands for exterior product. If we collect together the results just established we have finally arrived at important conclusion, that *by constructing matrix elements of operator-tensors of* $\hat{G}(2.2.3)$ *one produces the exterior products on wave manifold* $\widetilde{G}(2.2.3)$. There up on, *the matrix elements of symmetric operator-tensor identically equal zero.*
The linear operator-form of 1 degree $\hat{\omega}^1$ is a linear operator-valued function on $\hat{\mathbf{T}}_{\Phi_p}$, namely $\hat{\omega}^1(\hat{\mathbf{A}}_p) : \hat{\mathbf{T}}_{\Phi_p} \to \hat{R}$, where $\hat{\mathbf{A}}_p \in \hat{\mathbf{T}}_{\Phi_p}$, and the operator $\hat{\omega}^1(\hat{\mathbf{A}}) =< \hat{\omega}^1, \mathbf{A} >\in \hat{R}$ corresponded to $\hat{\mathbf{A}}_p$ at the point $\mathbf{\Phi}_p$, provided, according to eq.(3.43), with

$$< \chi \| \hat{\omega}^1 \| \chi^0 >= \sum_{\lambda,\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) \omega^1_{r^{\lambda\mu}}, \tag{4.7}$$

where $\omega^1_{r^{\lambda\mu}} = e^{(\lambda,\mu,\alpha)}_{r^{\lambda\mu}} \omega^{r^{\lambda\mu}}_{(\lambda,\mu,\alpha)}$, the $< \omega^1_{r^{\lambda\mu}}, \mathbf{A} >= \omega^1_{r^{\lambda\mu}}(\mathbf{A})$ is a linear form on $\mathbf{T}_p$, and

$$\begin{array}{l} \hat{\omega}^1(\lambda_1\hat{\mathbf{A}}_1 + \lambda_2\hat{\mathbf{A}}_2) = \lambda_1\hat{\omega}^1(\hat{\mathbf{A}}_1) + \lambda_2\hat{\omega}^1(\hat{\mathbf{A}}_2), \\ \forall\lambda_1, \lambda_2 \in R, \quad \hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2 \in \hat{\mathbf{T}}_{\Phi_p}. \end{array} \tag{4.8}$$

The set of all linear operator-forms at the point $\mathbf{\Phi}_p$ filled operator-vector space $\hat{\mathbf{T}}^*_{\Phi_p}$ being dual to $\hat{\mathbf{T}}_{\Phi_p}$. While, the $\{\hat{\gamma}^{(\lambda,\mu,\alpha)}_r\}$ served as a basis for them. The operator $n$-form may

be defined as the exterior product of operator 1-forms

$$\hat{\omega}^n(\hat{\mathbf{A}}_1, \dots, \hat{\mathbf{A}}_n) = (\hat{\omega}^1_1 \wedge \cdots \wedge \hat{\omega}^1_n)\left(\hat{\mathbf{A}}_1, \dots, \hat{\mathbf{A}}_n\right) =$$

$$= \left\| \begin{array}{ll} \hat{\omega}^1_1(\hat{\mathbf{A}}_1)\cdots\cdots & \hat{\omega}^1_n(\hat{\mathbf{A}}_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ \hat{\omega}^1_1(\hat{\mathbf{A}}_n)\cdots\cdots & \hat{\omega}^1_n(\hat{\mathbf{A}}_n) \end{array} \right\|. \tag{4.9}$$

Here, as well as for the rest of this section, to facilitate writing, we abbreviate the set of indices $(\lambda_i, \mu_i, \alpha_i)$ by the single symbol $i$. If $\{\hat{\gamma}^{r_i}_i\}$ and $\{\hat{\gamma}^i_{r_i}\}$ are dual basises respectively in $\hat{\mathbf{T}}_{\Phi_p}$ and $\hat{\mathbf{T}}^*_{\Phi_p}$, then the $\{\hat{\gamma}^{r_1}_1 \otimes \cdots \otimes \hat{\gamma}^{r_p}_p \otimes \hat{\gamma}^1_{s_1} \otimes \cdots \otimes \hat{\gamma}^q_{s_q}\}$ will be the basis in operator-space

$$\hat{\mathbf{T}}^p_q = \underbrace{\hat{\mathbf{T}}_{\Phi_p} \otimes \cdots \otimes \hat{\mathbf{T}}_{\Phi_p}}_{p} \otimes \underbrace{\hat{\mathbf{T}}^*_{\Phi_p} \otimes \cdots \otimes \hat{\mathbf{T}}^*_{\Phi_p}}_{q}. \tag{4.10}$$

Any operator-tensor $\hat{\mathbf{T}} \in \hat{\mathbf{T}}^p_q(\mathbf{\Phi}_p)$ can be written

$$\hat{\mathbf{T}} = T^{i_1\cdots i_p}_{j_1\cdots j_q}\,(r_1, \dots, r_p, s_1, \dots, s_q)\,\hat{\gamma}^{r_1}_{i_1} \otimes \cdots \otimes \hat{\gamma}^{r_p}_{i_p} \otimes \hat{\gamma}^{j_1}_{s_1} \otimes \cdots \otimes \hat{\gamma}^{j_q}_{s_q}, \tag{4.11}$$

where

$$T^{i_1\cdots i_p}_{j_1\cdots j_q}\,(r_1, \dots, r_p, s_1, \dots, s_q) = T\left(\hat{\gamma}^{i_1}_{r_1} \otimes \cdots \otimes \hat{\gamma}^{i_p}_{r_p} \otimes \hat{\gamma}^{s_1}_{j_1} \otimes \cdots \otimes \hat{\gamma}^{s_q}_{j_q}\right) \tag{4.12}$$

are the components of $\hat{\mathbf{T}}$ in dual basises $\{\hat{\gamma}^{r_i}_i\}$ and $\{\hat{\gamma}^i_{r_i}\}$. While, the conventional rules of tensor-algebra hold for them. In order to utilize the operator-tensor in its concrete expression, below according to eq.(3.43), explicitly the matrix element of operator-tensor of $\widehat{(p,q)}$-type may be written

$$\frac{1}{\sqrt{p!q!}} < \chi^0 \parallel \hat{T}^p_q \parallel \chi^0 >= \frac{1}{\sqrt{p!q!}} < \chi^0 \parallel T^{i_1\dots i_p}_{j_1\dots j_q}\hat{\gamma}^{r_1}_{i_1} \otimes \cdots \otimes \hat{\gamma}^{r_p}_{i_p} \otimes \hat{\gamma}^{j_1}_{s_1} \otimes \cdots \otimes \hat{\gamma}^{j_q}_{s_q} \parallel \chi^0 >=$$

$$= \left\{\sum_{\lambda,\mu=1}^{2}\right\}^p_1 \left\{\sum_{\tau,\nu=1}^{2}\right\}^q_1 \sum_{r^{\lambda\mu}_1,\dots,r^{\lambda\mu}_p=1}^{N_{\lambda\mu}} \sum_{s^{\tau\nu}_1,\dots,s^{\tau\nu}_q=1}^{N_{\tau\nu}} \bar{c}(r^{\lambda\mu}_1, \dots, r^{\lambda\mu}_p)\bar{c}^*(s^{\tau\nu}_1, \dots, s^{\tau\nu}_q)\times$$

$$\times T(r^{\lambda\mu}_1, \dots, r^{\lambda\mu}_p; s^{\tau\nu}_1, \dots, s^{\tau\nu}_q)^{\sigma(i_1\dots i_p)}_{\sigma(j_1\dots j_q)}(e^{r^{\lambda\mu}_1}_{i_1} \wedge \cdots \wedge e^{r^{\lambda\mu}_p}_{i_p}) \otimes (e^{j_1}_{s^{\tau\nu}_1} \wedge \cdots \wedge e^{j_q}_{s^{\tau\nu}_q}) =$$

$$= \left\{\sum_{\lambda,\mu=1}^{2}\right\}^p_1 \left\{\sum_{\tau,\nu=1}^{2}\right\}^q_1 \sum_{r^{\lambda\mu}_1,\dots,r^{\lambda\mu}_p=1}^{N_{\lambda\mu}} \sum_{s^{\tau\nu}_1,\dots,s^{\tau\nu}_q=1}^{N_{\tau\nu}} \bar{c}(r^{\lambda\mu}_1, \dots, r^{\lambda\mu}_p)\bar{c}^*(s^{\tau\nu}_1, \dots, s^{\tau\nu}_q)\times$$

$$\times T(r^{\lambda\mu}_1, \dots, r^{\lambda\mu}_p; s^{\tau\nu}_1, \dots, s^{\tau\nu}_q)^{\sigma(i_1\dots i_p)}_{\sigma(j_1\dots j_q)}(d\Phi^{r^{\lambda\mu}_1}_{i_1} \wedge \cdots \wedge d\Phi^{r^{\lambda\mu}_p}_{i_p}) \otimes (d\Phi^{j_1}_{s^{\tau\nu}_1} \wedge \cdots \wedge d\Phi^{j_q}_{s^{\tau\nu}_q}), \tag{4.13}$$

The matrix elements eq.(4.6) and eq.(4.13 ) are the geometric objects belonging to wave manifold $\widetilde{G}(2.2.3)$ exposing an antisymmetric part of tensor degree.
For any function $f : \mathbf{R}^n \to \mathbf{R}^n$ belonging to the ring of functions of $C^\infty$-class on $\hat{G}(2.2.3)$,

according to eq.(4.1), one defines an operator-differential by means of smooth reflection $\hat{d}\, f : \hat{\mathbf{T}}\left(\hat{G}(2.2.3)\right) \to \hat{R} \quad \left(\hat{\mathbf{T}}\left(\hat{G}(2.2.3)\right) = \bigcup_{\Phi_p} \hat{\mathbf{T}}_{\Phi_p}\right)$ as follows:

$$< \hat{d}\, f, \hat{\mathbf{A}} >= (Af)^{\wedge}, \tag{4.14}$$

where it is denoted

$$< \chi \| B^{\wedge} \| \chi^0 >= \sum_{\lambda,\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) B(r^{\lambda\mu}). \tag{4.15}$$

This rule for operator-differential is not without an important reason. The argument for this conclusion is compulsory suggested by the properties of the group of diffeomorphismes eq.(4.1). The ordinary laws regarding these changes apply that we ought to make use of

$$< \chi \| \hat{d}\, f, \hat{\mathbf{A}} \| \chi^0 >= \sum_{\lambda,\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) < d\, f, \mathbf{A} >_{r^{\lambda\mu}} = \sum_{\lambda,\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) (\mathbf{A}\, f)_{r^{\lambda\mu}}, \tag{4.16}$$

then in coordinate basis

$$< d\, \Phi^{\widehat{i}}, \hat{\partial}\, /\, \partial \Phi^j >= \frac{\partial \Phi^{\widehat{i}}}{\partial \Phi^j} = \delta^{\widehat{i}}_j, \tag{4.17}$$

provided $d\, \Phi^{\widehat{i}} \equiv \hat{d} \Phi^i$, the $\delta^{\widehat{i}}_j$ is being fashioned after the conventional symbol $\delta^i_j$ and best visualized as

$$< \chi \| \delta^{\widehat{i}}_j \| \chi^0 >= \sum_{\lambda,\mu=1}^{2} \sum_{r^{\lambda\mu}=1}^{N_{\lambda\mu}} \hat{c}^*(r^{\lambda\mu}) \delta^i_j, \quad (i \leftrightarrow (\lambda_i, \mu_i, \alpha_i)). \tag{4.18}$$

Continuing along this line we define the differential operator $n$-form $\hat{\omega}^n|_{\Phi_p}$ at the point $\mathbf{\Phi}_p \in \hat{G}(2.2.3)$ as the exterior operator $n$-form on tangent operator space $\hat{\mathbf{T}}_{\Phi_p}$ of tangent operator-vectors $\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n$. That is, if the $\wedge \hat{\mathbf{T}}^*_{\Phi_p}\left(\hat{G}(2.2.3)\right)$ means the exterior algebra on $\hat{\mathbf{T}}^*_{\Phi_p}\left(\hat{G}(2.2.3)\right)$, then operator $n$-form $\hat{\omega}^n|_{\Phi_p}$ is an element of $n$-th degree out of $\wedge \hat{\mathbf{T}}^*_{\Phi_p}$ depending on the point $\mathbf{\Phi}_p \in \hat{G}(2.2.3)$. Hence $\hat{\omega}^n = \bigcup_{\Phi_p} \hat{\omega}^n|_{\Phi_p}$. Any differential operator $n$-form belonging to dual operator-space $\underbrace{\hat{\mathbf{T}}^*_{\Phi_p} \otimes \cdots \otimes \hat{\mathbf{T}}^*_{\Phi_p}}_{n}$ may be written in form

$$\hat{\omega}^n = \sum_{i_1 < \cdots < i_n} \alpha_{i_1 \cdots i_n}(\Phi) d\, \Phi^{\widehat{i_1}} \wedge \cdots \wedge d\, \Phi^{\widehat{i_n}}, \tag{4.19}$$

provided by the smooth differentiable functions $\alpha_{i_1 \cdots i_n}(\Phi) \in C^\infty$ and basis

$$d\, \Phi^{\widehat{i_1}} \wedge \cdots \wedge d\, \Phi^{\widehat{i_n}} = \sum_{\sigma \in S_n} sgn(\sigma) \gamma^{\sigma(\widehat{i_1}} \otimes \cdots \otimes \gamma^{\widehat{i_n})}. \tag{4.20}$$

So, any antisymmetric operator-tensor of $\widehat{(0,n)}$-type reads

$$\hat{\mathbf{T}}^* = T_{i_1 \cdots i_n} \gamma^{\widehat{i_1}} \otimes \cdots \otimes \gamma^{\widehat{i_n}} = \sum_{i_1 < \cdots < i_n} T_{i_1 \cdots i_n} d\, \Phi^{\widehat{i_1}} \wedge \cdots \wedge d\, \Phi^{\widehat{i_n}}. \tag{4.21}$$

Let the $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ are two compact convex parallelepipeds in oriented $n$-dimensional operator-space $\hat{\mathbf{R}}^n$ and the $f : \hat{\mathcal{D}}_1 \to \hat{\mathcal{D}}_2$ is differentiable reflection of interior of $\hat{\mathcal{D}}_1$ into $\hat{\mathcal{D}}_2$ retaining an orientation, namely for any function $\varphi \in C^\infty$ defined on $\hat{\mathcal{D}}_2$ if holds $\varphi \circ f \in C^\infty$ and $f^*\varphi(\mathbf{\Phi}_p) = \varphi(f(\mathbf{\Phi}_p))$, where $f^*$ is an image of function $\varphi(f(\mathbf{\Phi}_p))$ on $\hat{\mathcal{D}}_1$ at the point $\mathbf{\Phi}_p$. Hence, the function $f$ induced a linear reflection $\hat{d} f : \hat{\mathbf{T}}\left(\hat{\mathcal{D}}_1\right) \to \hat{\mathbf{T}}\left(\hat{\mathcal{D}}_2\right)$ as an operator-differential of $f$ implying $\hat{d} f\left(\hat{\mathbf{A}}_p\right)\varphi = \hat{\mathbf{A}}_p(\varphi \circ f)$ for any operator-vector $\hat{\mathbf{A}}_p \in \hat{\mathbf{T}}_{\Phi_p}$ and for any function $\varphi \in C^\infty$ defined in the neighborhood of $\mathbf{\Phi}'_p = f(\mathbf{\Phi}_p)$. If the function $f$ is given in form $\Phi'^i = \Phi'^i\left(\Phi_p^{(1,1,1)}, \ldots, \Phi_p^{(2,2,3)}\right)$, where the coordinate suffixes were only put forth in illustration of a point at issue, and $\hat{\mathbf{A}}_p = \left(A^i\, \hat{\partial} \big/ \partial\Phi^i\right)_p$, then in terms of local coordinates one gets

$$\left(\hat{d} f\right)\hat{\mathbf{A}}_p = A^i \left(\frac{\partial \Phi'^j}{\partial \Phi^i}\right)_p \left(\frac{\hat{\partial}}{\partial \Phi'^j}\right)_{p'}. \tag{4.22}$$

So, if $f_1 : \hat{\mathcal{D}}_1 \to \hat{\mathcal{D}}_2$ and $f_2 : \hat{\mathcal{D}}_2 \to \hat{\mathcal{D}}_3$ then $\hat{d}\,(f_2 \circ f_1) = \hat{d} f_2 \circ \hat{d} f_1$. The differentiable reflection $f : \hat{\mathcal{D}}_1 \to \hat{\mathcal{D}}_2$ induced the reflection $\hat{f}^* : \hat{\mathbf{T}}^*\left(\hat{\mathcal{D}}_2\right) \to \hat{\mathbf{T}}^*\left(\hat{\mathcal{D}}_1\right)$ conjugated to $\hat{f}_*$. The latter is an operator differential of $f$, while

$$< \hat{f}^* \hat{\omega'}^1, \hat{\mathbf{A}} >_{\Phi_p} = < \hat{\omega'}^1, \hat{f}_* \hat{\mathbf{A}} > \Big|_{f(\Phi_p)}, \tag{4.23}$$

where $\hat{\mathbf{A}}\Big|_{f(\Phi_p)} = \left(\hat{d} f\right)\hat{\mathbf{A}}_p$ and $\hat{\omega'}^1 \in \hat{\mathbf{T}}^*\Big|_{f(\Phi_p)}$. Hence

$$\hat{f}^*\left(\hat{d}\,\varphi\right) = \hat{d}\,\left(\hat{f}^*\varphi\right) \tag{4.24}$$

and

$$\begin{aligned} &\hat{f}_* : \hat{\mathbf{T}} \in \hat{\mathbf{T}}^0_n(\Phi_p) \to \hat{f}_* \hat{\mathbf{T}} \in \hat{\mathbf{T}}^n_0(f(\Phi_p)), \\ &\hat{f}^* : \hat{\mathbf{T}} \in \hat{\mathbf{T}}^n_0(f(\Phi_p)) \to \hat{f}^* \hat{\mathbf{T}} \in \hat{\mathbf{T}}^0_n(\Phi_p). \end{aligned} \tag{4.25}$$

That is

$$\begin{aligned} &\hat{f}^*\, T\left(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n\right)\Big|_{\Phi_p} = T\left(\hat{f}_*\hat{\mathbf{A}}_1, \ldots, \hat{f}_*\hat{\mathbf{A}}_n\right)\Big|_{f(\Phi_p)}, \\ &T\left(\hat{f}^*\hat{\omega}^1_1, \ldots, \hat{f}^*\hat{\omega}^1_n\right)\Big|_{\Phi_p} = \hat{f}_*\, T\left(\hat{\omega}^1_1, \ldots, \hat{f}^*\hat{\omega}^1_n\right)\Big|_{f(\Phi_p)}. \end{aligned} \tag{4.26}$$

So, for any differential operator $n$-form $\hat{\omega}^n$ on $\hat{\mathcal{D}}_2$ the reflection $f$ induced the operator $n$-form $\hat{f}^*\hat{\omega}^n$ on $\hat{\mathcal{D}}_1$

$$\left(\hat{f}^*\hat{\omega}^n\right)\left(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n\right)\Big|_{\Phi_p} = \hat{f}_*\hat{\omega}^n\left(\hat{f}_*\hat{\mathbf{A}}_1, \ldots, \hat{f}_*\hat{\mathbf{A}}_n\right)\Big|_{f(\Phi_p)}. \tag{4.27}$$

If $\hat{\omega'}^1 = \alpha'_i d\,\Phi'^{\widehat{n}}$ then

$$\hat{f}^*\left(\alpha'_i d\,\Phi'^{\widehat{n}}\right) = \alpha'_i \frac{\partial \Phi'^i}{\partial \Phi^j} d\,\Phi^{\widehat{j}}. \tag{4.28}$$

This can be extended to $\hat{\omega'}^n \to \hat{\omega}^n$

$$\hat{f}^* \left( \sum_{i_1<\cdots<i_n} {T'}_{i_1\cdots i_n} d\,\Phi'^{\widehat{i_1}} \wedge \cdots \wedge d\,\Phi'^{\widehat{i_n}} \right) =$$
$$= \sum_{\substack{i_1<\cdots<i_n \\ j_1<\cdots<j_n}} {T'}_{i_1\cdots i_n} \frac{\partial \Phi'^{i_1}}{\partial \Phi^{j^1}} \cdots \frac{\partial \Phi'^{i_n}}{\partial \Phi^{j^n}} d\,\Phi^{\widehat{i_1}} \wedge \cdots \wedge d\,\Phi^{\widehat{i_n}}, \tag{4.29}$$

namely

$$\hat{f}^* \hat{\omega'}^n = J_\Phi \hat{\omega}^n = (det\, df)\, \hat{\omega}^n, \tag{4.30}$$

where $J_\Phi$ is the Jacobian of reflection $J_\Phi = \left\| \dfrac{\partial \Phi'^i}{\partial \Phi^j} \right\|$. While, the following relations hold

$$\left(\hat{f}_1 \circ \hat{f}_2\right)^* = \hat{f}_1^* \circ \hat{f}_2^*, \quad \hat{f}^* \left(\hat{\omega}_1 \wedge \hat{\omega}_2\right) = \hat{f}^* \left(\hat{\omega}_1\right) \wedge \hat{f}^* \left(\hat{\omega}_2\right). \tag{4.31}$$

We may consider the integration of operator $n$-form implying

$$\int_{\hat{\mathcal{D}}_1} \hat{f}^* \hat{\omega}^n = \int_{\hat{\mathcal{D}}_2} \hat{\omega}^n. \tag{4.32}$$

In general, let the $\hat{\mathcal{D}}_1$ is a limited convex $n$-dimensional parallelepiped in $n$-dimensional operator space $\hat{\mathbf{R}}^n$. One defined the $n$-dimensional $i$-th piece of integration path $\hat{\sigma}^i$ in $\hat{G}(2.2.3)$ as $\hat{\sigma}^i = \left(\hat{\mathcal{D}}_i, f_i, Or_i\right)$, where $\hat{\mathcal{D}}_i \in \hat{\mathbf{R}}^n$, $\quad f_i : \hat{\mathcal{D}}_i \to \hat{G}(2.2.3)$ and the $Or_i$ is an orientation of $\hat{\mathbf{R}}^n$. Then, the integral over the operator $n$-form $\hat{\omega}^n$ along the operator $n$-dimensional chain $\hat{c}_n = \sum m_i \hat{\sigma}^i$ may be written

$$\int_{\hat{c}_n} \hat{\omega}^n = \sum m_i \int_{\hat{\sigma}^i} \hat{\omega}^n = \sum m_i \int_{\hat{\mathcal{D}}_i} \hat{f}^* \hat{\omega}^n, \tag{4.33}$$

where the $m_i$ is a multiple number. Taking into account the eq.(4.7), its matrix element yields

$$< \chi \| \int_{\hat{c}_n} \hat{\omega}^n \| \chi^0 > \to \left\{ \sum_{\lambda\mu=1}^{2} \right\}_1^n \sum_{r_1^{\lambda\mu},\ldots,r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \sum m_i \bar{c}(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) \int_{\hat{\mathcal{D}}_i} \hat{f}^* \hat{\omega}^n (r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}). \tag{4.34}$$

Next we employ the analog of exterior differentiation adjusted to fit a discussing formalism. So, we may define the value of the operator $(n+1)$-form $\hat{d}\,\hat{\omega}^n$ on $(n+1)$ operator-vectors $\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_{n+1} \in \hat{\mathbf{T}}_{\Phi_p}$ by considering diffeomorphic reflection $f$ of the neighborhood of the point 0 in $\hat{\mathbf{R}}^n$ into neighborhood of the point $\mathbf{\Phi}_p$ in $\hat{G}(2.2.3)$. The prototypes of operator-vectors $\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_{n+1} \in \hat{\mathbf{T}}_{\Phi_p}\left(\hat{G}(2.2.3)\right)$ at the operator differential of $f$ belong to tangent operator space $\hat{\mathbf{R}}^n$ in 0. Then, the prototypes are the operator-vectors $\hat{\xi}_1, \ldots, \hat{\xi}_{n+1} \in \hat{\mathbf{R}}^n$. Let $f$ reflects the parallelepiped $\hat{\mathbf{\Pi}}^*$, stretched over the $\hat{\xi}_1, \ldots, \hat{\xi}_{n+1}$, into $(n+1)$- dimensional piece $\hat{\mathbf{\Pi}}$ on $\hat{G}(2.2.3)$. While the border of $n$-dimensional chain $\partial\hat{\mathbf{\Pi}}$ in $\hat{\mathbf{R}}^{n+1}$ defined as follows: the pieces $\hat{\sigma}^i$ of the chain $\partial\hat{\mathbf{\Pi}}$ are $n$-dimensional facets $\partial\hat{\mathbf{\Pi}}_i$ of parallelepiped $\partial\hat{\mathbf{\Pi}}$ with the reflections $f_i : \hat{\mathbf{\Pi}}_i \to \hat{\mathbf{R}}^{n+1}$ of embedding of the facets into $\hat{\mathbf{R}}^{n+1}$, and the

orientations $Or_i$ defined $\partial\hat{\mathbf{\Pi}} = \sum \hat{\sigma}^i, \quad \hat{\sigma}^i = \left(\hat{\mathbf{\Pi}}_i, f_i, Or_i\right)$ Considering the curvilinear parallelepiped

$$F\left(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n\right) = \int_{\partial\hat{\mathbf{\Pi}}} \hat{\omega}^n, \tag{4.35}$$

one may state that the unique operator of $(n+1)$-form $\hat{\Omega}$ exists on $\hat{\mathbf{T}}_{\Phi_p}$, which is the major $(n+1)$-linear part in 0 of integral over the border of $F\left(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_n\right)$, namely

$$F\left(\varepsilon\hat{\mathbf{A}}_1, \ldots, \varepsilon\hat{\mathbf{A}}_n\right) = \varepsilon^{n+1}\hat{\Omega}\left(\hat{\mathbf{A}}_1, \ldots, \hat{\mathbf{A}}_{n+1}\right) + O\left(\varepsilon^{n+1}\right), \tag{4.36}$$

where the $\hat{\Omega}$ is independent of the choice of coordinates used in definition of $F$. A prove of it may be readily furnished in analogy with the same statement of corresponding theorem of differential geometry [9]. If in local coordinates

$$\hat{\omega}^n = \sum_{i_1 < \cdots < i_n} T_{i_1 \cdots i_n} d\,\Phi^{\widehat{i_1}} \wedge \cdots \wedge d\,\Phi^{\widehat{i_n}}, \tag{4.37}$$

then

$$\hat{\Omega} = \hat{d}\,\hat{\omega}^n = \sum_{i_1 < \cdots < i_n} \hat{d}\,T_{i_1 \cdots i_n} d\,\Phi^{\widehat{i_1}} \wedge \cdots \wedge d\,\Phi^{\widehat{i_n}}. \tag{4.38}$$

The operator of exterior differential $\hat{d}$ commutes with the reflection $f : \hat{G}(2.2.3) \to \hat{G}(2.2.3)$

$$\hat{d}\left(\hat{f}^*\hat{\omega}^n\right) = \hat{f}^*\left(\hat{d}\,\hat{\omega}^n\right). \tag{4.39}$$

So define the exterior differential by operator-form $d\hat{\omega}$ of $(n+1)$ degree

$$\begin{aligned} \hat{d}\hat{\omega}^n = & \sum_{\substack{i_0 \\ i_1 < \ldots < i_n}} \frac{\partial T_{i_1 \ldots i_n}}{\partial \Phi^{i_0}} d\Phi^{\widehat{i_0}} \wedge d\Phi^{\widehat{i_1}} \wedge \cdots \wedge d\Phi^{\widehat{i_n}} = \\ & = \sum_{i_1 < \ldots < i_n} (\hat{d}T_{i_1 \ldots i_n}) \wedge d\Phi^{\widehat{i_1}} \wedge \cdots \wedge d\Phi^{\widehat{i_n}}, \end{aligned} \tag{4.40}$$

then

$$\begin{aligned} & < \chi \parallel \hat{d}\hat{\omega}^n \parallel \chi^0 > \longrightarrow \\ & \sum_{\substack{i_0 \\ i_1 < \ldots < i_n}} \left\{\sum_{\lambda,\mu=1}^{2}\right\}_0^n \sum_{r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_0^{\lambda\mu}, r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}) \times \\ & \times \frac{\partial T(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu})_{i_1 \ldots i_n}}{\partial \Phi^{i_0}_{r_0^{\lambda\mu}}} d\Phi^{i_0}_{r_0^{\lambda\mu}} \wedge d\Phi^{i_1}_{r_1^{\lambda\mu}} \wedge \cdots \wedge d\Phi^{i_n}_{r_n^{\lambda\mu}} = \\ & = \sum_{i_1 < \ldots < i_n} \left\{\sum_{\lambda,\mu=1}^{2}\right\}_1^n \sum_{r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu}=1}^{N_{\lambda\mu}} \bar{c}(r_1^{\lambda\mu}, \ldots r_n^{\lambda\mu})(dT(r_1^{\lambda\mu}, \ldots, r_n^{\lambda\mu})_{i_1 \ldots i_n}) \wedge d\Phi^{i_1}_{r_1^{\lambda\mu}} \wedge \cdots \wedge d\Phi^{i_n}_{r_n^{\lambda\mu}}. \end{aligned} \tag{4.41}$$

In generalized sense, we may draw a conclusion that *the matrix element of any geometric object of operator manifold* $\hat{G}(2.2.3)$ *yields corresponding geometric object of wave manifold* $\widetilde{G}(2.2.3)$. Thus, *all geometric objects belonging to the latter can be constructed by means of matrix elements of corresponding geometric objects of the operator manifold* $\hat{G}(2.2.3)$.

One final observation is worth recording. We may consider a linear differential operator $\hat{L}_{\hat{\mathbf{A}}}$, namely the differentiation along the direction of operator-vector $\hat{\mathbf{A}}$. For any function $\varphi \in C^\infty, \quad \varphi : \hat{G}(2.2.3) \to R$ a derivative along a direction of $\hat{\mathbf{A}}$ is an operator-function $\hat{L}_{\hat{\mathbf{A}}}\varphi$ taking at the point $\mathbf{\Phi}_p$ the value

$$\left(\hat{L}_{\hat{\mathbf{A}}}\varphi(\mathbf{\Phi})\right) = \left.\frac{\hat{d}}{d\,t}\right|_{t=0} \varphi\left(A^t \mathbf{\Phi}\right). \tag{4.42}$$

In local coordinates $\{\Phi^i\}$ the flux $A^t$ is given by equation $\hat{\dot{\Phi}}^i(\zeta) = \widehat{A^i}(\zeta)$. So

$$\hat{L}_{\hat{\mathbf{A}}} = A^i \frac{\hat{\partial}}{\partial \Phi^i}. \tag{4.43}$$

Let it be $\hat{\mathbf{A}}, \hat{\mathbf{B}} \in \hat{\mathbf{T}}_{\Phi_p}\left(\hat{G}(2.2.3)\right)$. Then, two fluxes $A^t$ and $B^t$ are commuting only if Poison bracket $[\hat{\mathbf{A}}, \hat{\mathbf{B}}]$ is zero, while for the field $\hat{\mathbf{C}} = [\hat{\mathbf{A}}, \hat{\mathbf{B}}]$ it holds $\hat{L}_{\hat{\mathbf{C}}} = \hat{L}_{\hat{\mathbf{B}}}\hat{L}_{\hat{\mathbf{A}}} - \hat{L}_{\hat{\mathbf{A}}}\hat{L}_{\hat{\mathbf{B}}}$. We may extend this up to differentiation of operator-tensor $\hat{\mathbf{T}}$ at the point $\mathbf{\Phi}_p$ along the direction of $\hat{\mathbf{A}}$

$$\hat{L}_{\hat{\mathbf{A}}}\hat{\mathbf{T}}\Big|_{\Phi_p} = \lim_{t\to 0}\frac{1}{t}\left\{\hat{\mathbf{T}}\Big|_{\Phi_p} - \hat{f}_{t^*}\,\hat{\mathbf{T}}\Big|_{\Phi_p}\right\}. \tag{4.44}$$

We hope that we have made good headway by presenting a reasonable analysis of mathematical structure of the method of operator manifold $\hat{G}(2.2.3)$.

# 5 Discussion and Conclusions

At this point we cut short our exposition of the theory and reflect upon the results far obtained. A number of conclusions may be drawn and the main features of suggested theory are outlined below. The method of operator manifold is a still wider generalization of secondary quantization with appropriate expansion over the geometric objects, leading to the quantization of geometry differed in principle from all earlier suggested schemes. It has two important aspects of quantum field theory and differential geometry. We first deal with a substitution of the basis vectors of tangent section of the manifold $G(2.2.3)$ by corresponding operators of creation and annihilation of quanta of geometry acting in the configuration space of occupation numbers. While, as it was shown, the quantum of geometry may be regarded as a fermion field described by the theory being in close analogy to Dirac's conventional wave-mechanical theory of fermions with a spin $\frac{\vec{1}}{2}$ treated in terms of manifold $G(2.2.3)$. The final formulation of the method is mainly based on configuration space mechanics with antisymmetric state functions incorporated with geometric properties of corresponding elements. In view of this, one has reached to well-grounded rigorous definition of concept of operator manifold $\hat{G}(2.2.3)$. Completing the quantum field aspect of operator manifold $\hat{G}(2.2.3)$, we have constructed the explicit forms

of wave state functions and calculated the matrix elements of concrete field operators. The differential geometric aspect, which is a second interesting offshoot of the method of operator manifold, is discussed in detail in last section. We defined the operator-tensors and have drawn an important conclusion, that the matrix elements of operator-tensors of $\hat{G}(2.2.3)$ yield the external product on wave manifold $\widetilde{G}(2.2.3)$. One considered also the differential operator forms and employed their integration as well as exterior differentiation, the differentiation along the direction of operator-vector, adjusted to fit the formalism of operator manifold. One may state that the matrix element of any geometric object of operator manifold $\hat{G}(2.2.3)$ yields corresponding geometric object of wave manifold $\widetilde{G}(2.2.3)$.

Acknowledgements

I am greatly indebted to A.M.Vardanian and K.L.Yerknapetian for support.

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